1 Information matrix for the model with endogenous regressors

The log-likelihood function for the model in level is

$$\begin{aligned} l(\alpha, \gamma', \sigma_{\mathrm{m}}^2) &= c - \frac{N}{2} \ln\left(\delta_{11} - \delta_{*1}'\delta_{**}^{-1}\delta_{*1}\right) - \sum_{i=1}^{N} \frac{s_{iT}^2}{2\Delta_{11.*}} \\ &= c + \sum_{i=1}^{N} l_i, \end{aligned}$$

where $l_i = (-\frac{1}{2})\left[\ln\left(\delta_{11} - \delta_{*1}'\delta_{**}^{-1}\delta_{*1}\right) + \frac{s_{iT}^2}{\Delta_{11.*}}\right]$,
$\delta_{11} = \sigma_{\mathrm{u}}^2 \sum_{j=0}^{T-2} \alpha^{2j} + (1-\alpha^{T-1})^2\sigma_{\mathrm{m}}^2$, $\delta_{21} = (1-\alpha^{T-1})\phi_{pm}$,
$\delta_{13} = (1-\alpha^{T-1})\sigma_{\mathrm{m}}^2 + (1-\alpha^{T-1})\gamma'\phi_{pm}$,
$\delta_{23} = \phi_{pm} + \phi_{pp}\gamma, \delta_{33} = \sigma_{\mathrm{m}}^2 + \gamma'\phi_{pp}\gamma + \sigma_{\mathrm{x}_1}^2 + 2\gamma'\phi_{pm}$,
$\Delta_{11.*} = \delta_{11} - \delta_{*1}'\delta_{**}^{-1}\delta_{*1}$, and $s_{iT} = y_{iT} - (1-\alpha^{T-1})\gamma' p_i - \alpha^{T-1} y_{i1} - \delta_{*1}'\delta_{**}^{-1} r_i$

Let δ_{33},δ_{32}, and $\lambda = Var(y_{i2} - y_{i1}) = \sigma_{\mathrm{u}}^2 + (1-\alpha)^2\sigma_{\mathrm{x}_1}^2$ be given.

Then,

$$\begin{aligned} \sigma_{\mathrm{u}}^2 &= \lambda - (1-\alpha)^2\sigma_{\mathrm{x}_1}^2; \\ \phi_{pm} &= \delta_{23} - \phi_{pp}\gamma; \\ \sigma_{\mathrm{x}_1}^2 &= \delta_{33} - \sigma_{\mathrm{m}}^2 + \gamma'\phi_{pp}\gamma - 2\gamma'\delta_{23}; \end{aligned}$$

Thus,

$$\begin{aligned} \delta_{11} &= \left(\lambda - (1-\alpha)^2\sigma_{\mathrm{x}_1}^2\right) \sum_{j=0}^{T-2} \alpha^{2j} + (1-\alpha^{T-1})^2\sigma_{\mathrm{m}}^2 \\ &= \left(\lambda - (1-\alpha)^2(\delta_{33} - \sigma_{\mathrm{m}}^2 + \gamma'\phi_{pp}\gamma - 2\gamma'\delta_{23})\right) \sum_{j=0}^{T-2} \alpha^{2j} + (1-\alpha^{T-1})^2\sigma_{\mathrm{m}}^2; \\ \delta_{21} &= (1-\alpha^{T-1})(\delta_{23} - \phi_{pp}\gamma) \\ \delta_{13} &= (1-\alpha^{T-1})\sigma_{\mathrm{m}}^2 + (1-\alpha^{T-1})\gamma'(\delta_{23} - \phi_{pp}\gamma) \end{aligned}$$

(i) $\frac{\partial l_i}{\partial \alpha}$

Since

$$
\begin{aligned}
\partial\delta_{11}/\partial\alpha &= 2\{(1-\alpha)(\delta_{33}-\sigma_{\mathrm{m}}^2+\gamma'\phi_{pp}\gamma-2\gamma'\delta_{23})\sum_{j=0}^{T-2}\alpha^{2j} \\
&\quad +\left(\lambda-(1-\alpha)^2(\delta_{33}-\sigma_{\mathrm{m}}^2+\gamma'\phi_{pp}\gamma-2\gamma'\delta_{23})\right)\sum_{j=0}^{T-2}j\alpha^{2j-1} \\
&\quad -(T-1)\alpha^{T-2}(1-\alpha^{T-1})\sigma_{\mathrm{m}}^2\}; \\
\partial\delta_{21}/\partial\alpha &= -(T-1)\alpha^{T-2}(\delta_{23}'-\gamma'\phi_{pp}); \\
\partial\delta_{13}/\partial\alpha &= -(T-1)\alpha^{T-2}\sigma_{\mathrm{m}}^2-(T-1)\alpha^{T-2}\gamma'(\delta_{23}-\phi_{pp}\gamma); \\
\partial\Delta_{11.*}/\partial\alpha &= \partial\delta_{11}/\partial\alpha-2\left[\partial\delta_{21}/\partial\alpha \quad \partial\delta_{13}/\partial\alpha\right]\delta_{**}^{-1}\delta_{*1}; \\
&: = \Delta_{11.*}^{d\alpha} \\
\partial s_{iT}/\partial\alpha &= (T-1)\alpha^{T-2}\gamma' p_i-(T-1)\alpha^{T-2}y_{i1}-\left[\partial\delta_{21}/\partial\alpha \quad \partial\delta_{13}/\partial\alpha\right]\delta_{**}^{-1}q_i \\
&= (T-1)\alpha^{T-2}\left[\gamma' \quad -1\right]q_i-\left[\partial\delta_{21}/\partial\alpha \quad \partial\delta_{13}/\partial\alpha\right]\delta_{**}^{-1}q_i \\
&= \left\{(T-1)\alpha^{T-2}\left[\gamma' \quad -1\right]-\left[\partial\delta_{21}/\partial\alpha \quad \partial\delta_{13}/\partial\alpha\right]\delta_{**}^{-1}\right\}q_i \\
&: = s_{iT}^{d\alpha},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\frac{\partial l_i}{\partial\alpha} &= -\frac{1}{2}\frac{\Delta_{11.*}^{d\alpha}}{\Delta_{11.*}}-\frac{1}{2}\frac{1}{(\Delta_{11.*})^2}\left\{2s_{iT}(\partial s_{iT}/\partial\alpha)\Delta_{11.*}-s_{iT}^2\left(\partial\Delta_{11.*}/\partial\alpha\right)\right\} \\
&= \left(-\frac{1}{2}\right)\left[\frac{\Delta_{11.*}^{d\alpha}}{\Delta_{11.*}}+\frac{2s_{iT}s_{iT}^{d\alpha}}{\Delta_{11.*}}-\frac{s_{iT}^2\Delta_{11.*}^{d\alpha}}{(\Delta_{11.*})^2}\right].
\end{aligned}
$$

(ii) $E\left(\frac{\partial l_i}{\partial\alpha}\right)^2$

Note that

$$E(s_{iT}s_{iT}^{d\alpha})=0, \tag{1}$$

which implies that s_{iT} and $s_{iT}^{d\alpha}$ are independent. In addition,

$$E\left(s_{iT}^4\right)=3\left(\Delta_{11.*}\right)^2 \tag{2}$$

and let

$$c_\alpha' := (T-1)\alpha^{T-2}\left[\gamma' \quad -1\right]-\left[\partial\delta_{21}/\partial\alpha \quad \partial\delta_{13}/\partial\alpha\right]\delta_{**}^{-1}$$

Relations (1) and (2) yield

$$
\begin{aligned}
E(\frac{\partial l_i}{\partial \alpha})^2 &= \frac{1}{4}\{\left(\frac{\Delta_{11.*}^{d\alpha}}{\Delta_{11.*}}\right)^2 - 2\left(\frac{\Delta_{11.*}^{d\alpha}}{\Delta_{11.*}}\right)\left(\frac{E(s_{iT}^2)\Delta_{11.*}^{d\alpha}}{\left(\Delta_{11.*}\right)^2}\right) + \frac{E\left(s_{iT}^4\right)\left(\Delta_{11.*}^{d\alpha}\right)^2}{\left(\Delta_{11.*}\right)^4} + \frac{4E(s_{iT}^2)E(s_{iT}^{d\alpha 2})}{(\Delta_{11.*})^2}\} \\
&= \frac{1}{4}\left[\left(\frac{\Delta_{11.*}^{d\alpha}}{\Delta_{11.*}}\right)^2 - 2\left(\frac{\Delta_{11.*}^{d\alpha}}{\Delta_{11.*}}\right)\left(\frac{\Delta_{11.*}^{d\alpha}}{\Delta_{11.*}}\right) + \frac{3\left(\Delta_{11.*}^{d\alpha}\right)^2}{\left(\Delta_{11.*}\right)^2} + \frac{4E(s_{iT}^{d\alpha 2})}{\Delta_{11.*}}\right] \\
&= \frac{\left(\Delta_{11.*}^{d\alpha}\right)^2}{2\left(\Delta_{11.*}\right)^2} + \frac{E(s_{iT}^{d\alpha 2})}{\Delta_{11.*}} \\
&= \frac{\left(\Delta_{11.*}^{d\alpha}\right)^2}{2\left(\Delta_{11.*}\right)^2} + \frac{c_\alpha' \delta_{**} c_\alpha}{\Delta_{11.*}}
\end{aligned}
$$

(iii) $\frac{\partial l_i}{\partial \gamma}$

$$
\begin{aligned}
\partial\delta_{11}/\partial\gamma &= -(1-\alpha)^2(2\phi_{pp}\gamma - 2\delta_{23})\sum_{j=0}^{T-2}\alpha^{2j}; \\
\partial\delta_{21}/\partial\gamma &= -(1-\alpha^{T-1})\phi_{pp}; \\
\partial\delta_{13}/\partial\gamma &= (1-\alpha^{T-1})\delta_{23} - 2(1-\alpha^{T-1})\phi_{pp}\gamma; \\
\partial\Delta_{11.*}/\partial\gamma &= \partial\delta_{11}/\partial\gamma - 2\ \ \partial\delta_{21}/\partial\gamma \quad \partial\delta_{13}/\partial\gamma\ \ \delta_{**}^{-1}\delta_{*1}; \\
&: = \Delta_{11.*}^{d\gamma} \\
\partial s_{iT}/\partial\gamma &= -(1-\alpha^{T-1})p_i - \ \ \partial\delta_{21}/\partial\gamma \quad \partial\delta_{13}/\partial\gamma\ \ \delta_{**}^{-1}q_i \\
&= \{-(1-\alpha^{T-1})\ \ I_{l_p} \quad 0 \ \ - \ \ \partial\delta_{21}/\partial\gamma \quad \partial\delta_{13}/\partial\gamma\ \ \delta_{**}^{-1}\}q_i \\
&: = s_{iT}^{d\gamma}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\frac{\partial l_i}{\partial \gamma} &= -\frac{1}{2\Delta_{11.*}}\Delta_{11.*}^{d\gamma} - \frac{1}{2}\frac{2s_{iT}}{\Delta_{11.*}}s_{iT}^{d\gamma} + \frac{1}{2}\frac{s_{iT}^2}{(\Delta_{11.*})^2}\Delta_{11.*}^{d\gamma} \\
&= \left(-\frac{1}{2}\right)\left[\frac{\Delta_{11.*}^{d\gamma}}{\Delta_{11.*}} + \frac{2s_{iT}}{\Delta_{11.*}}s_{iT}^{d\gamma} - \frac{s_{iT}^2\Delta_{11.*}^{d\gamma}}{(\Delta_{11.*})^2}\right]
\end{aligned}
$$

Let

$$B_\gamma' := -(1-\alpha^{T-1})\ \ I_{l_p} \quad 0 \ \ - \ \ \partial\delta_{21}/\partial\gamma \quad \partial\delta_{13}/\partial\gamma\ \ \delta_{**}^{-1}$$

(iv) $E(\frac{\partial l_i}{\partial \gamma}\frac{\partial l_i}{\partial \gamma'})$

we have

$$
\begin{aligned}
E(\frac{\partial l_i}{\partial \gamma}\frac{\partial l_i}{\partial \gamma'}) &= \frac{1}{4}E\left[\frac{\Delta_{11.*}^{d\gamma}}{\Delta_{11.*}} + \frac{2s_{iT}^{d\gamma}s_{iT}}{\Delta_{11.*}} - \frac{s_{iT}^2\Delta_{11.*}^{d\gamma}}{(\Delta_{11.*})^2}\right]\left[\frac{(\Delta_{11.*}^{d\gamma})'}{\Delta_{11.*}} + \frac{2s_{iT}(s_{iT}^{d\gamma})'}{\Delta_{11.*}} - \frac{(\Delta_{11.*}^{d\gamma})'s_{iT}^2}{(\Delta_{11.*})^2}\right] \\
&= \frac{1}{4}[\frac{(\Delta_{11.*}^{d\gamma})(\Delta_{11.*}^{d\gamma})'}{(\Delta_{11.*})^2} - \frac{(\Delta_{11.*}^{d\gamma})(\Delta_{11.*}^{d\gamma})'}{(\Delta_{11.*})^2} + \frac{4E[s_{iT}^2]E[s_{iT}^{d\gamma}(s_{iT}^{d\gamma})']}{(\Delta_{11.*})^2} \\
&\quad -\frac{\Delta_{11.*}^{d\gamma}(\Delta_{11.*}^{d\gamma})'}{(\Delta_{11.*})^2} + \frac{3(\Delta_{11.*}^{d\gamma})(\Delta_{11.*}^{d\gamma})'}{(\Delta_{11.*})^2}] \\
&= \frac{(\Delta_{11.*}^{d\gamma})(\Delta_{11.*}^{d\gamma})'}{2(\Delta_{11.*})^2} + \frac{B_\gamma'\delta_{**}B_\gamma}{\Delta_{11.*}}
\end{aligned}
$$

(v) $E(\frac{\partial l_i}{\partial \gamma})\left(\frac{\partial l_i}{\partial \alpha}\right)$

$$
\begin{aligned}
E(\frac{\partial l_i}{\partial \gamma})\left(\frac{\partial l_i}{\partial \alpha}\right) &= \left(\frac{1}{4}\right)E\left[\frac{\Delta_{11.*}^{d\gamma}}{\Delta_{11.*}} + \frac{2s_{iT}^{d\gamma}s_{iT}}{\Delta_{11.*}} - \frac{s_{iT}^2\Delta_{11.*}^{d\gamma}}{(\Delta_{11.*})^2}\right]\left[\frac{\Delta_{11.*}^{d\alpha}}{\Delta_{11.*}} + \frac{2s_{iT}s_{iT}^{d\alpha}}{\Delta_{11.*}} - \frac{s_{iT}^2\Delta_{11.*}^{d\alpha}}{(\Delta_{11.*})^2}\right] \\
&= \left(\frac{1}{4}\right)\left[\frac{\Delta_{11.*}^{d\gamma}\Delta_{11.*}^{d\alpha}}{(\Delta_{11.*})^2} - \frac{\Delta_{11.*}^{d\gamma}\Delta_{11.*}^{d\alpha}}{(\Delta_{11.*})^2} + \frac{4E[s_{iT}^{d\gamma}s_{iT}^{d\alpha}]}{\Delta_{11.*}} - \frac{\Delta_{11.*}^{d\gamma}\Delta_{11.*}^{d\alpha}}{(\Delta_{11.*})^2} + \frac{3\Delta_{11.*}^{d\gamma}\Delta_{11.*}^{d\alpha}}{(\Delta_{11.*})^2}\right] \\
&= \frac{E[s_{iT}^{d\gamma}s_{iT}^{d\alpha}]}{\Delta_{11.*}} + \frac{\Delta_{11.*}^{d\gamma}\Delta_{11.*}^{d\alpha}}{2(\Delta_{11.*})^2} \\
&= \frac{\Delta_{11.*}^{d\alpha}\Delta_{11.*}^{d\gamma}}{2(\Delta_{11.*})^2} + \frac{B_\gamma'\delta_{**}c_\alpha}{\Delta_{11.*}}
\end{aligned}
$$

(vi) $\frac{\partial l}{\partial \sigma_{\mathrm{m}}^2}$

Since

$$
\begin{aligned}
\partial\delta_{11}/\partial\sigma_{\mathrm{m}}^2 &= (1-\alpha)^2\sum_{j=0}^{T-2}\alpha^{2j} + (1-\alpha^{T-1})^2; \\
\partial\delta_{21}/\partial\sigma_{\mathrm{m}}^2 &= 0; \\
\partial\delta_{13}/\partial\sigma_{\mathrm{m}}^2 &= (1-\alpha^{T-1}); \\
\partial\Delta_{11.*}/\partial\sigma_{\mathrm{m}}^2 &= \partial\delta_{11}/\partial\sigma_{\mathrm{m}}^2 - 2\ \ \partial\delta_{21}/\partial\sigma_{\mathrm{m}}^2 \quad \partial\delta_{31}/\partial\sigma_{\mathrm{m}}^2\ \ \delta_{**}^{-1}\delta_{*1}; \\
&: = \Delta_{11.*}^{dm} \\
\partial s_{iT}/\partial\sigma_{\mathrm{m}}^2 &= -\ \ \partial\delta_{21}/\partial\sigma_{\mathrm{m}}^2 \quad \partial\delta_{13}/\partial\sigma_{\mathrm{m}}^2\ \ \delta_{**}^{-1}q_i \\
&: = s_{iT}^{dm}
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\frac{\partial l_i}{\partial\sigma_{\mathrm{m}}^2} &= -\frac{1}{2}\frac{\Delta_{11.*}^{dm}}{\Delta_{11.*}} - \frac{1}{2}\frac{2s_{iT}^{dm}s_{iT}}{\Delta_{11.*}} + \frac{1}{2}\frac{s_{iT}^2\Delta_{11.*}^{dm}}{(\Delta_{11.*})^2} \\
&= \left(-\frac{1}{2}\right)\left[\frac{\Delta_{11.*}^{dm}}{\Delta_{11.*}} + \frac{2s_{iT}^{dm}s_{iT}}{\Delta_{11.*}} - \frac{s_{iT}^2\Delta_{11.*}^{dm}}{(\Delta_{11.*})^2}\right]
\end{aligned}
$$

(vii) $E(\frac{\partial l_i}{\partial \sigma^2_{\mathrm{m}}})^2$
Let

$$\begin{aligned} E(\frac{\partial l_i}{\partial \sigma^2_{\mathrm{m}}})^2 &= \frac{1}{4}E\left[\frac{\Delta^{dm}_{11.*}}{\Delta_{11.*}} + \frac{2s^{dm}_{iT}s_{iT}}{\Delta_{11.*}} - \frac{s^2_{iT}\Delta^{dm}_{11.*}}{(\Delta_{11.*})^2}\right]^2 \\ &= \frac{1}{4}\left[\frac{(\Delta^{dm}_{11.*})^2}{(\Delta_{11.*})^2} + \frac{4E[(s^{dm}_{iT})^2]}{\Delta_{11.*}} + \frac{3(\Delta^{dm}_{11.*})^2}{(\Delta_{11.*})^2} - \frac{2(\Delta^{dm}_{11.*})^2}{(\Delta_{11.*})^2}\right] \\ &= \frac{(\Delta^{dm}_{11.*})^2}{2(\Delta_{11.*})^2} + \frac{E[(s^{dm}_{iT})^2]}{\Delta_{11.*}} \end{aligned}$$

(viii) $E\left(\frac{\partial l_i}{\partial \alpha}\frac{\partial l_i}{\partial \sigma^2_{\mathrm{m}}}\right)$

$$\begin{aligned} & E\left(\frac{\partial l_i}{\partial \alpha}\frac{\partial l_i}{\partial \sigma^2_{\mathrm{m}}}\right) \\ = \;& E\left(-\frac{1}{2}\right)\left[\frac{\Delta^{d\alpha}_{11.*}}{\Delta_{11.*}} + \frac{2s_{iT}s^{d\alpha}_{iT}}{\Delta_{11.*}} - \frac{s^2_{iT}\Delta^{d\alpha}_{11.*}}{(\Delta_{11.*})^2}\right]\left(-\frac{1}{2}\left[\frac{\Delta^{dm}_{11.*}}{\Delta_{11.*}} + \frac{2s^{dm}_{iT}s_{iT}}{\Delta_{11.*}} - \frac{s^2_{iT}\Delta^{dm}_{11.*}}{(\Delta_{11.*})^2}\right]\right) \\ = \;& \frac{1}{4}\left[\frac{\Delta^{d\alpha}_{11.*}\Delta^{dm}_{11.*}}{(\Delta_{11.*})^2} - \frac{\Delta^{d\alpha}_{11.*}\Delta^{dm}_{11.*}}{(\Delta_{11.*})^2} + \frac{4E[s^{d\alpha}_{iT}s^{dm}_{iT}]}{\Delta_{11.*}} - \frac{\Delta^{d\alpha}_{11.*}\Delta^{dm}_{11.*}}{(\Delta_{11.*})^2} + \frac{3\Delta^{d\alpha}_{11.*}\Delta^{dm}_{11.*}}{(\Delta_{11.*})^2}\right] \\ = \;& \frac{\Delta^{d\alpha}_{11.*}\Delta^{dm}_{11.*}}{2(\Delta_{11.*})^2} + \frac{E[s^{d\alpha}_{iT}s^{dm}_{iT}]}{\Delta_{11.*}} \end{aligned}$$

(ix) $E\left(\frac{\partial l_i}{\partial \gamma}\frac{\partial l_i}{\partial \sigma^2_{\mathrm{m}}}\right)$

$$\begin{aligned} & E\left(\frac{\partial l_i}{\partial \gamma}\frac{\partial l_i}{\partial \sigma^2_{\mathrm{m}}}\right) \\ = \;& \frac{1}{4}E\left[\frac{\Delta^{dm}_{11.*}}{\Delta_{11.*}} + \frac{2s^{dm}_{iT}s_{iT}}{\Delta_{11.*}} - \frac{s^2_{iT}\Delta^{dm}_{11.*}}{(\Delta_{11.*})^2}\right]\left[\frac{\Delta^{d\gamma}_{11.*}}{\Delta_{11.*}} + \frac{2s^{d\gamma}_{iT}s_{iT}}{\Delta_{11.*}} - \frac{s^2_{iT}\Delta^{d\gamma}_{11.*}}{(\Delta_{11.*})^2}\right] \\ = \;& \frac{1}{4}\left[\frac{\Delta^{dm}_{11.*}\Delta^{d\gamma}_{11.*}}{(\Delta_{11.*})^2} - \frac{\Delta^{dm}_{11.*}\Delta^{d\gamma}_{11.*}}{(\Delta_{11.*})^2} + \frac{4E[s^{dm}_{iT}s^{d\gamma}_{iT}]}{\Delta_{11.*}} - \frac{\Delta^{dm}_{11.*}\Delta^{d\gamma}_{11.*}}{(\Delta_{11.*})^2} + \frac{3\Delta^{dm}_{11.*}\Delta^{d\gamma}_{11.*}}{(\Delta_{11.*})^2}\right] \\ = \;& \frac{\Delta^{dm}_{11.*}\Delta^{d\gamma}_{11.*}}{2(\Delta_{11.*})^2} + \frac{E[s^{dm}_{iT}s^{d\gamma}_{iT}]}{\Delta_{11.*}} \end{aligned}$$