Chapter 5. Multivariate Volatility Models

In Choi

Sogang University

Motivation

- See Chapter 9 of Tsay, and Bauwens, L. et al. (2006) "Multivariate GARCH models: a survey", J. Appl. Econ. 21: 79–109.
- Study of the relations between the volatilities and co-volatilities of several markets.
- Is the volatility of a market leading the volatility of other markets?
- Is the volatility of an asset transmitted to another asset directly (through its conditional variance) or indirectly (through its conditional covariances)?

Motivation

- Does a shock on a market increase the volatility on another market, and by how much?
- Is the impact the same for negative and positive shocks of the same amplitude?
- Do the correlations between asset returns change over time? Are they higher during periods of higher volatility? Are they increasing in the long run, perhaps because of the globalization of financial markets?

Basic model

- $\{r_t\}$ in a k x 1 multivariate return series.
- Write

$$r_t = \mu_t + a_t$$

where

$$\begin{aligned} \mu_t &= E(r_t \mid F_{t-1}), \\ a_t &= (a_{1t}, ..., a_{kt})', \end{aligned}$$

and

$$a_t = \Sigma_t \varepsilon_t \text{ with } E\varepsilon_t = 0 \text{ and } E\varepsilon_t \varepsilon_t' = I_k.$$

Basic model

- Assume a vecter ARMA structure for μ_t. That is,

$$\mu_t = \phi_0 + \sum_{i=1}^{p} \Phi_i r_{t-i} - \sum_{i=1}^{q} \Theta_i a_{t-i}.$$

- $\Sigma_t = Cov(a_t \mid F_{t-1})$: conditional variance of a_t.
  we should have $\Sigma_t > 0$.
  There are $\frac{k(k+1)}{2}$ unknown parameters in Σ_t.
- There are various models for Σ_t.
  These constitute multivariate volatility models.

- See Bollerslev, Engle and Wooldridge (1992, JPE).
- Let $h_t = vech(\Sigma_t)$ where $vech(.)$ denotes the operator that stacks the lower triangular portion of k x k matrix as a $\frac{k(k+1)}{2}$ x 1 vector.

Example If $\Sigma_t = \begin{bmatrix} \sigma_{11t} & \sigma_{12t} \\ \sigma_{12t} & \sigma_{22t} \end{bmatrix}$, then $h_t = \begin{bmatrix} \sigma_{11t} \\ \sigma_{12t} \\ \sigma_{22t} \end{bmatrix}$.

- In addition, let $\eta_t = vech(a_t a_t')$.
- The vec(1,1) model is defined as

$$h_t = C + A\eta_{t-1} + Gh_{t-1}.$$

Example If $k = 2$, the model becomes

$$\begin{bmatrix} \sigma_{11t} \\ \sigma_{12t} \\ \sigma_{22t} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} a_{1,t-1}^2 \\ a_{1,t-1}a_{2,t-1} \\ a_{2,t-1}^2 \end{bmatrix} + \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11,t-1} \\ \sigma_{12,t-1} \\ \sigma_{22,t-1} \end{bmatrix}.$$

There are 21 unknown parameters for this model.

- In order to reduce the number of parameters, Bollerslev et al. suggest the diagonal VEC (DVEC) model. Here, A and G are diagonal matrices.

Example If $k = 2$,

$$\begin{bmatrix} \sigma_{11t} \\ \sigma_{12t} \\ \sigma_{22t} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} a_{1,t-1}^2 \\ a_{1,t-1} a_{2,t-1} \\ a_{2,t-1}^2 \end{bmatrix} + \begin{bmatrix} G_{11} & 0 & 0 \\ 0 & G_{22} & 0 \\ 0 & 0 & G_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11,t-1} \\ \sigma_{12,t-1} \\ \sigma_{22,t-1} \end{bmatrix}.$$

There are only 9 unknown parameters for this model.

- Define the symmetric k x k matrices A^0, G^0, C^0 by the relations

$$A = diag(vech(A^0))$$
$$G = diag(vech(G^0))$$

and

$$C = vech(C^0).$$

- For example, when $k = 2$,

$$A^0 = \begin{bmatrix} A_{11} & A_{22} \\ A_{22} & A_{33} \end{bmatrix}, \; G^0 = \begin{bmatrix} G_{11} & G_{22} \\ G_{22} & G_{33} \end{bmatrix}, \; C^0 = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix}.$$

- The DVEC model can be written as

$$\Sigma_t = C^0 + A^0 \odot (a_{t-1} a'_{t-1}) + G^0 \odot \Sigma_{t-1}$$

where $X \odot Y$ is the matrix containing elementwise products (x_{ij}, y_{ij}).
- Σ_t is positive definite for all t provided that C^0, A^0, G^0 and H^0 are positive definite.
- A special case of the DVEC model assumes that σ_{ijt} $(i \neq j)$ do not vary over time, and introduces dynamics only for $\sigma_{11t}, \sigma_{22t}, \ldots, \sigma_{kkt}$. This model is called the constant-correlation model.

- Letting $\Sigma_t^* = (\sigma_{11t}, \sigma_{22t}, ..., \sigma_{kkt})$, this model is written as

$$\Sigma_t^* = C^* + A^* a_{t-1} \odot a_{t-1} + G^* \Sigma_{t-1}^*.$$

Example Let $k = 2$. Then,

$$\begin{aligned} \begin{bmatrix} \sigma_{11t} \\ \sigma_{22t} \end{bmatrix} &= \begin{bmatrix} c_1^* \\ c_2^* \end{bmatrix} + \begin{bmatrix} A_{11}^* & A_{12}^* \\ A_{21}^* & A_{22}^* \end{bmatrix} \begin{bmatrix} a_{1t-1}^2 \\ a_{2t-1}^2 \end{bmatrix} \\ &+ \begin{bmatrix} G_{11}^* & G_{12}^* \\ G_{21}^* & G_{22}^* \end{bmatrix} \begin{bmatrix} \sigma_{11,t-1} \\ \sigma_{12,t-1} \end{bmatrix}. \end{aligned}$$

- The constant-correlation model can be written as

$$\begin{aligned} a_t \odot a_t &= C^* + (A^* + G^*) a_{t-1} \odot a_{t-1} + \gamma_t - G^* \gamma_{t-1}, \\ \gamma_t &= a_t \odot a_t - \Sigma_t^*. \end{aligned}$$

- This is a VARMA(1,1) model for $a_t \odot a_t$.

Example Monthly log returns of IBM stock and the S&P 500 index from January 1926 to December 1999 $(r_{1t} \& r_{2t})$

$$\begin{aligned} r_{1t} &= 1.351 + 0.072 r_{1,t-1} + 0.055 r_{1,t-2} - 0.119 r_{2,t-2} + a_{1t} \\ r_{2r} &= 0.703 + a_{2t} \end{aligned}$$

The volatility equations are

$$\begin{aligned} \begin{bmatrix} \sigma_{11,t} \\ \sigma_{22,t} \end{bmatrix} &= \begin{bmatrix} 2.98 \\ 2.09 \end{bmatrix} + \begin{bmatrix} 0.079 & 0 \\ 0.042 & 0.045 \end{bmatrix} \begin{bmatrix} a^2_{1t-1} \\ a^2_{2t-1} \end{bmatrix} \\ &+ \begin{bmatrix} 0.873 & -0.031 \\ -0.066 & 0.913 \end{bmatrix} \begin{bmatrix} \sigma_{11,t-1} \\ \sigma_{12,t-1} \end{bmatrix}. \end{aligned}$$

This shows a feedback relationship between the volatility of the two monthly log returns.

- See Engle and Kroner (1995, Econometric Theory).
- Motivation: Strong restrictions on the parameters are required to guarantee that Σ_t be positive.
- The BEKK(1,1,K) model is defined as

$$\Sigma_t = C'C + \sum_{i=1}^{K} A_i' a_{t-1} a_{t-1}' A_i + \sum_{j=1}^{K} G_j' \Sigma_{t-1} G_j$$

where C, A_k, G_k are $k \times k$ matrices and C is upper triangular.

GARCH models for multivariate returns

BEKK model

- The BEKK model is a special case of the VEC model. Consider the case $k = 2, \;\; K = 1$

$$
\begin{aligned}
\begin{bmatrix} \sigma_{11t} & \sigma_{12t} \\ \sigma_{12t} & \sigma_{22t} \end{bmatrix} \;\; = \;\; & \begin{bmatrix} c_{11} & 0 \\ c_{12} & c_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{bmatrix} \\
& + \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} a_{1,t-1}^2 & a_{1,t-1} a_{2,t-1} \\ a_{1,t-1} a_{2,t-1} & a_{2,t-1}^2 \end{bmatrix} \left[\begin{matrix} A_{11} & A_{1} \\ A_{21} & A_{2} \end{matrix} \right. \\
& + \begin{bmatrix} G_{11} & G_{21} \\ G_{12} & G_{22} \end{bmatrix} \begin{bmatrix} \sigma_{11,t-1} & \sigma_{12,t-1} \\ \sigma_{12,t-1} & \sigma_{22,t-1} \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}
\end{aligned}
$$

- Thus,

$$\begin{bmatrix} \sigma_{11t} \\ \sigma_{12t} \\ \sigma_{22t} \end{bmatrix} = C^* + A^* \begin{bmatrix} a_{1,t-1}^2 \\ a_{1,t-1} a_{2,t-1} \\ a_{2,t-1}^2 \end{bmatrix} + G^* \begin{bmatrix} \sigma_{11,t-1} \\ \sigma_{12,t-1} \\ \sigma_{22,t-1} \end{bmatrix},$$

where C^*, A^* and G^* are determined by the equation above.

- One may impose diagonal structures for A_i and G_j to reduce the number of parameters. This model is guaranteed to be positive definite while the DVEC is not.
- The VEC and BEKK models are rarely used when the dimension of the series is larger than 3 or 4 due to the large number of the parameters of the models.

- See Engle, Ng, Rothschild (1990, JOE).
- Σ_t is parameterized by using the idea that co-movements of the stock returns are driven by a small number of common underlying variables, called factors.
- Assume

$$a_t = \Lambda f_t + e_t$$

  where Λ is an k x l matrix, f_t is an l x 1 vector, and e_t is an idiosyncratic shock with constant variance matrix and uncorrelated with f_t.

- Assume

$$E(f_{it} \mid F_{t-1}) = 0$$

$$Var(f_{it} \mid F_{t-1}) = \sigma^2_{f_{it}}$$

and

$$\sigma^2_{f_{it}} = \omega_i + \alpha_i^2 f^2_{i,t-1} + \beta_i^2 \sigma^2_{f_i,t-1}.$$

- Consider the case where $l = 1$. Then,

$$\begin{aligned} a_t &= \lambda f_t + e_t \\ Var(a_{it} &\mid F_{t-1}) = Var(\lambda_i f_t \mid F_{t-1}) + Var(e_{it}) \\ &= \lambda_i \sigma^2_{f_t} + \sigma^2_e \end{aligned}$$

and

$$\sigma^2_{f_t} = \omega + \alpha^2 f^2_{t-1} + \beta^2 \sigma^2_{f,t-1}.$$

- In practice,
  (1) Estimate factors by the pricipal component analysis.
  (2) Build a volatility model for the estimated factors.
  (3) Relate the volatility of the factors to that of $\{a_t\}$.