Homework 7

Mathematical Statistics (Fall, 2018)
Total points: 10
Due date: 12. 3 (M)

1. Let $X_1, ..., X_n$ be iid $Bernoulli(p)$. Show that the LRT of $H_0 : p \leq p_0$ vs. $H_1 : p > p_0$ will reject H_0 if $\sum_{i=1}^n Y_i > b$.

2. Let $X_1, ..., X_n$ be a random sample from the pdf

$$f(x \mid \theta) = \frac{1}{\lambda} \exp(-\frac{(x-\theta)}{\lambda}) \mathbf{1}(\theta \leq x < \infty).$$

   Find the LRT of $H_0 : \theta \leq 0$ vs. $H_1 : \theta > 0$.

3. Show that for a random sample $X_1, ..., X_n$ from a $N(0, \sigma^2)$ population, the most powerful test of $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$, where $\sigma_0 < \sigma_1$, is given by

$$\phi(\sum_{i=1}^n X_i^2) = \{ \begin{array}{ll} 1 & \text{if } \sum_{i=1}^n X_i^2 > c \\ 0 & \text{if } \sum_{i=1}^n X_i^2 \leq c \end{array}$$

   For a given value of α, the size of the Type I Error, show how the value of c is explicitly determined.

4. Let $X_1, ..., X_{n+1}$ be iid $Bernoulli(p)$.

   (a) Find the most powerful test of size $\alpha = .0547$ of the hypotheses $H_0 : p = 1/2$ vs. $H_1 : p = 1/4$. Find the power of this test.

   (b) For testing $H_0 : p \leq 1/2$ vs. $H_1 : p > 1/2$, find the size and sketch the power function of the test that rejects H_0 if $\sum_{i=1}^{10} X_i \geq 6$.

   (c) For what α levels does there exist a UMP test of the hypotheses in part (a)?

5. Let $f(x \mid \theta)$ be the logistic location pdf

$$f(x \mid \theta) = \frac{e^{(x-\theta)}}{(1 + e^{(x-\theta)})^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

   (a) Show that this family has an MLR.

   (b) Based on one observation, X, find the most powerful size α test of $H_0 : \theta = 0$ vs. $H_1 : \theta = 1$. For $\alpha = 0.2$, find the size of the Type II error.

   (c) Show that the test in part (b) is UMP size α for testing $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$.

6. Let $X_1, ..., X_n$ be iid $Poisson(\lambda)$.

   (a) Find a UMP test of $H_0 : \lambda \leq \lambda_0$ vs. $H_1 : \lambda > \lambda_0$.

(b) Consider the specific case $H_0 : \lambda \leq 1$ vs. $H_1 : \lambda > 1$. Use the CLT to determine the sample size n so a UMP test satisfies $P(reject\ H_0 \mid \lambda = 1) = 0.05$ and $P(reject\ H_0 \mid \lambda = 2) = 0.9$.

7. Let $X_1, ..., X_n$ be iid $N(\mu_X, \sigma_X^2)$ and let $Y_1, ..., Y_n$ be iid $N(\mu_Y, \sigma_Y^2)$. We are interested in testing

$$H_0 : \mu_X = \mu_Y \text{ vs. } H_1 : \mu_X \neq \mu_Y$$

with the assumption $\sigma_X^2 = \sigma_Y^2 = \sigma^2$.

(a) Show that the LRT can be based on the statistic

$$T = \frac{\bar{X} - \bar{Y}}{\sqrt{S_p^2(\frac{1}{n} + \frac{1}{m})}},$$

where

$$S_p^2 = \frac{1}{n+m-2}\left(\sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{i=1}^{m}(Y_i - \bar{Y})^2.\right)$$

(b) Show that $T \sim t_{n+m-2}$ under H_0.