Homework 6

Mathematical Statistics (Fall, 2018)
Total points: 10
Due date: 11. 26 (M)

1. Let $X_1, ..., X_n$ be a random sample from the pdf

$$f(x \mid \theta) = \theta x^{-2},\ 0 < \theta \le x < \infty.$$

   (a) What is a sufficient statistic for θ?
   (b) Find the MLE of θ.
   (c) Show that the method of moments estimator of θ does not exist. [Hint: Derive EX.]

2. One observation, X, is taken from a $N(0, \sigma^2)$ population.

   (a) Find an unbiased estimator of σ^2.
   (b) Find the MLE of σ.
   (c) Discuss how the method of moments estimator of σ might be found.

3. Let $X_1, ..., X_n$ be a random sample from the pdf

$$f(x \mid \theta) = \frac{1}{\theta},\ 0 \le x \le \theta,\ \theta > 0.$$

   Estimate θ using both the method of moments and maximum likelihood. Calculate the means and variances of the two estimators. Which one should be preferred and why?

4. Suppose that the random variables $Y_1, ..., Y_n$ satisfy

$$Y_i = \beta x_i + \varepsilon_i,\ i = 1, ..., n,$$

   where $x_1, ..., x_n$ are fixed constants, and $\varepsilon_i \sim iid\ N(0, \sigma^2)$.

   (a) Find a two-dimensional sufficient statistic for (β, σ^2).
   (b) Find the MLE of β, and show that it is an unbiased estimator of β.
   (c) Find the distribution of the MLE of β.
   (d) Show that $\sum y_i / \sum x_i$ is an unbiased estimator of β.
   (e) Calculate the variance of $\sum y_i / \sum x_i$ and compare it to that of MLE.

5. Let $X_1, ..., X_n$ be iid $Bernoulli(p)$.

   (a) Show that the variance of $\bar{X}$ attains the Cramer-Rao Lower Bound, and hence $\bar{X}$ is the best unbiased estimator of p.

(b) For $n > 4$, show that the product $X_1X_2X_3X_4$ is an unbiased estimator of p^4. Using Rao-Blackwell theorem, construct a better estimator of p^4 than $X_1X_2X_3X_4$. [Hint: The estimator is a conditional mean of $X_1X_2X_3X_4$.]

6. Let $X_1, ..., X_{n+1}$ be iid $Bernoulli(p)$, and define $h(p)$ as

$$h(p) = P\left(\sum_{i=1}^{n} X_i > X_{n+1} \mid p\right).$$

(a) Show that $T(X_1, ..., X_{n+1}) = \mathbf{1}(\sum_{i=1}^{n} X_i > X_{n+1})$ is an unbiased estimator of $h(p)$.

(b) Using Rao-Blackwell theorem, construct a better estimator of $h(p)$ than $T(X_1, ..., X_{n+1})$.

7. Let $\{X_i\}_{i=1,...,n}$ be a sequence of independent random variables with the common pdf $f(x \mid \theta)$. Consider an estimator of $\tau(\theta)$, $T_n(X_1, ..., X_n)$. Suppose that

$$ET_n = \tau(\theta) + \sum_{i=1}^{K} a_i(\theta)/n^i,$$

where K is a constant greater than 2 and unrelated to n and $a_i(\theta)$ is a function of θ unrelated to n. This relation shows that T_n is biased for $\tau(\theta)$ in finite samples and unbiased for $\tau(\theta)$ asymptotically. Let $T_{-i,n-1} = T_{n-1}(X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$ and $\bar{T}_{n-1} = \frac{1}{n}\sum_{i=1}^{n} T_{-i,n-1}$. Now consider a new statistic $T_n' = nT_n - (n-1)\bar{T}_{n-1}$. This estimator is called the jackknife estimator. Show that $n(ET_n - \tau(\theta)) \to c_1$ $(const.)$ and that $n^2(ET_n' - \tau(\theta)) \to c_2$ $(const.)$. These show that the bias of T_n' converges to zero faster than that of T_n.

8. Let $\{X_i\}_{i=1,...,n}$ be a sequence of independent random variables having the Bernoulli distribution with parameter θ. We are interested in estimating θ^2. A natural estimator for this is $T_n = \left(\frac{\sum_{i=1}^{n} X_i}{n}\right)^2$.

(a) Show that T_n is biased for θ^2 in finite samples.

(b) Show that the jackknife estimator of θ^2 is unbiased.