Econometrics for Financial Time Series

Chapter 3: Multiple Time Series Analysis

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Multiple Time Series Analysis

- Reference:
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Weak stationarity and cross-correlation matrix

- Let $r_t = \begin{pmatrix} r_{1t} \\ \vdots \\ r_{Kt} \end{pmatrix}$.
- Mean vector:

$$\mu_t = E(r_t) = \begin{pmatrix} E(r_{1t}) \\ \vdots \\ E(r_{Kt}) \end{pmatrix} = \begin{pmatrix} \mu_{1t} \\ \vdots \\ \mu_{Kt} \end{pmatrix}$$

- Covariance matrices

$$\Gamma_{tl} = Cov(r_t, r_{t-l}) = E[(r_t - \mu_t)(r_{t-l} - \mu_t)'] = [\Gamma_{tij}(l)].$$

Weak stationarity and cross-correlation matrix

- Notice that Γ_{tl} is not a symmetric matrix when $l \neq 0$. When $l = 0$,

$$\begin{aligned}
\Gamma_{t0} &= E[(r_t - \mu_t)(r_t - \mu_t)'] \\
&= \begin{bmatrix} E[(r_{1t} - \mu_{1t})(r_{1t} - \mu_{1t})] & \cdots & E[(r_{1t} - \mu_{1t})(r_{kt} - \mu_{kt})] \\ \vdots & \ddots & \vdots \\ E[(r_{kt} - \mu_{1t})(r_{1t} - \mu_{1t})] & \cdots & E[(r_{kt} - \mu_{kt})(r_{kt} - \mu_{kt})] \end{bmatrix} \\
&= [\Gamma_{tij}(0)].
\end{aligned}$$

  The diagonal elements are variances and off-diagonal elements covariances.

- The multivariate time series $\{r_t\}$ is said to be (weakly) stationary if μ_t and Γ_{tl} are independent of the time index t.

Weak stationarity and cross-correlation matrix

- Assume $\{r_t\}$ is stationary. Let

$$D = diag[\sqrt{\Gamma_{11}(0)}, ..., \sqrt{\Gamma_{kk}(0)}].$$

The concurrent cross-correlation matrix (CCM) of r_t is defined as

$$\rho_0 = D^{-1}\Gamma_0 D^{-1} = [\rho_{ij}(0)].$$

The (i, j)th elements of ρ_0 is the correlation between r_{it} and r_{jt}.

$$\rho_{ij}(0) = \frac{\Gamma_{ij}(0)}{\sqrt{\Gamma_{ii}(0)}\sqrt{\Gamma_{jj}(0)}} = \frac{Cov(r_{it}, r_{jt})}{std(r_{it})std(r_{jt})}.$$

Weak stationarity and cross-correlation matrix

- The lag$-l$ cross-correlation matrix of r_t is defined by

$$\rho_l = D^{-1}\Gamma_l D^{-1} = [\rho_{ij}(l)].$$

$\rho_{ij}(l)$ is the correlation between r_{it} and $r_{j,t-l}$.
Since

$$\begin{aligned}\Gamma_{ij}(l) &= Cov(r_{it}, r_{j,t-l}) \\ &= Cov(r_{j,t-l}, r_{it}) \\ &= Cov(r_{j,t}, r_{i,t+l}) \text{ (stationarity)} \\ &= Cov(r_{j,t}, r_{i,t-(-l)}) \\ &= \Gamma_{ji}(-l),\end{aligned}$$

we have

$$\Gamma_l = \Gamma'_{-l}.$$

Weak stationarity and cross-correlation matrix

1. r_{it} and r_{jt} have no linear relationship if $\rho_{ij}(l) = \rho_{ji}(l) = 0$ for all $l \geq 0$.
2. r_{it} and r_{jt} are concurrently correlated if $\rho_{ij}(0) \neq 0$.
3. r_{it} and r_{jt} have no lead-lag relationship if $\rho_{ij}(l) = \rho_{ji}(l) = 0$ for all $l > 0$.
4. There is a unidirectional relationship from r_{it} to r_{jt} if $\rho_{ij}(l) = 0$ for all $l > 0$, but $\rho_{ji}(v) \neq 0$ for some $v > 0$. (r_{jt} depends on some past values of r_{it}).
5. There is a feedback relationship between r_{it} and r_{jt} if $\rho_{ij}(l) \neq 0$ for some $l > 0$ and $\rho_{ji}(v) \neq 0$ for some $v > 0$.

Weak stationarity and cross-correlation matrix

- Sample cross-correlation matrixes

$$\begin{aligned} \hat{\Gamma}_l &= \frac{1}{T}\sum_{t=l+1}^{T}(r_t-\bar{r})(r_{t-l}-\bar{r})',\ l \geq 0, \\ \bar{r} &= \frac{1}{T}\sum_{t=1}^{T} r_t. \end{aligned}$$

$$\hat{\rho}_l = \hat{D}^{-1}\hat{\Gamma}_l\hat{D}^{-1},\ l\ \geq 0.$$

- Multivariate Ljung–Box test

$$Q_K(m) = T^2\sum_{l=1}^{m}\frac{1}{T-l}tr(\hat{\Gamma}_l'\hat{\Gamma}_0^{-1}\hat{\Gamma}_l\hat{\Gamma}_0^{-1}) \sim \chi^2(K^2m).$$

- VAR(1) model

$$r_t = \phi_0 + \Phi r_{t-1} + a_t,$$

where ϕ_0 a k-dimensional vector, Φ is a $K \times K$ matrix, and $\{a_t\}$ is a sequence of serially uncorrelated random vectors with mean zero and covariance matrix Σ.

- Bivariate case

$$
\begin{aligned}
r_{1t} &= \phi_{10} + \Phi_{11} r_{1,t-1} + \Phi_{12} r_{2,t-1} + a_{1t} \\
r_{2t} &= \phi_{20} + \Phi_{21} r_{1,t-1} + \Phi_{22} r_{2,t-1} + a_{2t}
\end{aligned}
$$

Φ_{12} : linear dependence of r_{1t} on $r_{2,t-1}$ in the presence of $r_{1,t-1}$
Φ_{21} : linear dependence of r_{2t} on $r_{1,t-1}$ in the presence of $r_{2,t-1}$
$\Phi_{12} = 0$ and $\Phi_{21} \neq 0$: a unidirectional relationship from r_{1t} to r_{2t}
$\Phi_{12} = 0$ and $\Phi_{21} = 0$: r_{1t} and r_{2t} are uncoupled.
$\Phi_{12} \neq 0$ and $\Phi_{21} \neq 0$: a feedback relationship between r_{1t} and r_{2t}

- The concurrent relationship between r_{1t} and r_{2t} is shown by the off-diagonal element σ_{12} of the covariance matrix Σ.

- There exists a lower triangular matrix L with all of its diagonal elements being equal to one such that $\Sigma = LGL'$ where G is a diagonal matrix.
  Define $b_t = L^{-1}a_t$. Then,

$$E(b_t) = 0, \; Cov(b_t) = L^{-1}\Sigma(L^{-1})' = G$$

and

$$\begin{aligned} L^{-1}r_t &= L^{-1}\phi_0 + L^{-1}\Phi r_{t-1} + b_t \\ &= \phi_0^* + \Phi^* r_{t-1} + b_t. \end{aligned}$$

- The $j-$th equation of this model is

$$r_{jt} + \sum_{i=1}^{j-1} \omega_{ji} r_{it} = \phi_{j,0}^{*} + \sum_{i=1}^{j} \Phi_{ji}^{*} r_{i,t-1} + b_{jt},$$

where ω_{ji} are the elements of the j-th row of L. This shows explicitly the concurrent linear dependence of r_{jt} on $r_{1t}, \ldots, r_{j-1,t}$.

- Assume that the VAR(1) model is weakly stationary. Since

$$E(r_t) = \phi_0 + \Phi E(r_{t-1}),$$

$$\mu = E(r_t) = (I - \Phi)^{-1}\phi_0.$$

Using $\phi_0 = (I - \Phi)\mu$, write

$$r_t - \mu = \Phi(r_{t-1} - \mu) + a_t$$

or

$$\tilde{r}_t = \Phi\tilde{r}_{t-1} + a_t.$$

Repeated substitutions give

$$\tilde{r}_t = a_t + \Phi a_{t-1} + \Phi^2 a_{t-2} + \ldots.$$

1.

$$Cov(a_t, r_{t-1}) = 0.$$

2.

$$Cov(a_t, r_t) = \Sigma.$$

3. Φ^j must converge to zero as $j \longrightarrow \infty$. Otherwise, Φ^j will either explode or converge to a nonzero matrix as $j \longrightarrow \infty$.

4. For Φ^j to converge to zero as $j \longrightarrow \infty$, all eigenvalues of Φ should be less than 1 in modulus. In fact, this is the condition for the stationarity of r_t.
5.

$$E(\tilde{r}_t \tilde{r}'_{t-l}) = \Phi E(\tilde{r}_{t-1} \tilde{r}'_{t-l})$$

or

$$\Gamma_l = \Phi \Gamma_{l-1}, \ l > 0.$$

This gives

$$\Gamma_l = \Phi^l \Gamma_0, \ l > 0.$$

- VAR(p) model

$$r_t = \phi_0 + \Phi_1 r_{t-1} + \ldots + \Phi_p r_{t-p} + a_t.$$

Assume that the VAR(p) model is weakly stationary. Since

$$E(r_t) = \phi_0 + \Phi_1 E(r_{t-1}) + \ldots + \Phi_p E(r_{t-p}),$$

$$\mu = E(r_t) = (I - \Phi_1 - \ldots - \Phi_p)^{-1}\phi_0.$$

Using $\phi_0 = (I - \Phi_1 - \ldots - \Phi_p)\mu$, write

$$r_t - \mu = \Phi_1(r_{t-1} - \mu) + \ldots + \Phi_p(r_{t-p} - \mu) + a_t$$

or

$$\tilde{r}_t = \Phi_1 \tilde{r}_{t-1} + \ldots + \Phi_p \tilde{r}_{t-p} + a_t.$$

1. $$Cov(a_t, r_{t-l}) = 0 \text{ for } l > 0.$$
2. $$Cov(a_t, r_t) = \Sigma.$$
3. $$\Gamma_l = \Phi_1 \Gamma_{l-1} + ... + \Phi_p \Gamma_{l-p}, \ \ l > 0.$$

- The VAR(p) model can be written as the VAR(1) model
  Let

$$x_t = \begin{bmatrix} \tilde{r}_{t-p+1} \\ \tilde{r}_{t-p+2} \\ \vdots \\ \tilde{r}_t \end{bmatrix} \text{ and } b_t = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_t \end{bmatrix}.$$

- Then, the VAR(p) model can be written as

$$x_t = \Phi^* x_{t-1} + b_t,$$

where

$$\Phi^* = \begin{bmatrix} 0 & I & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \\ \Phi_p & \Phi_{p-1} & \Phi_{p-2} & \Phi_{p-3} & \cdots & \Phi_1 \end{bmatrix}.$$

- Note that the last row of Φ^*signifies the VAR(p) model and that the rest are identity relations. This representation tells that if all eigenvalues of Φ^* are less than 1 in modulus, r_t is weakly stationary. But this is equivalent to

$$|I - \Phi_1 z - \cdots - \Phi_p z^p| \neq 0 \text{ for } |z| \leq 1.$$

Estimating the VAR(p) model

- $vec(\cdot)$ operator: Let $\underset{m\times n}{A} = (a_1 \cdots a_n)$. Then,

$$vec(A) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \cdot {}_{mn\times 1 \ vector}$$

Estimating the VAR(p) model

Example

If

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

$$vec(A) = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}.$$

Estimating the VAR(p) model

Definition

The Kronecker product

Let

$$\underset{m\times n}{A} = (a_{ij}) \text{ and } \underset{p\times q}{B} = (b_{ij}).$$

The $mp \times nq$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

is the Kronecker product of A and B.

Estimating the VAR(p) model

Example

Let
$$A = \begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix}$$
and
$$B = \begin{bmatrix} 4 & 5 \end{bmatrix}.$$
Then,
$$A \otimes B = \begin{bmatrix} 3[4\ 5] & 2[4\ 5] \\ 1[4\ 5] & 7[4\ 5] \end{bmatrix} = \begin{bmatrix} 12 & 15 & 8 & 10 \\ 4 & 5 & 28 & 35 \end{bmatrix}.$$

Estimating the VAR(p) model

- The following property of the $vec(\cdot)$ operator will be useful.

$$vec(AB) = (B' \otimes I)vec(A).$$

- Write the $VAR(p)$ model

$$r_t = \mu + \Phi_1 r_{t-1} + \cdots + \Phi_p r_{t-p} + a_t$$

as a multivariate linear regression model

$$Y = BW + U$$

Estimating the VAR(p) model

where

$$Y = (r_1, \cdots, r_n)$$

$$B = (\mu, \Phi_1, \cdots, \Phi_p)$$

$$\begin{aligned} W &= (W_0, \cdots, W_{n-1}) \\ U &= (a_1, \cdots, a_n) \end{aligned}$$

and

$$W_t = \begin{bmatrix} \mathbf{1} \\ r_t \\ \vdots \\ r_{t-p+1} \end{bmatrix},$$

where $\mathbf{1} = [1, ..., 1]'$.

Estimating the VAR(p) model

- Using the $vec(\cdot)$ operator, the $VAR(p)$ model can be written compactly as

$$\begin{aligned} vec(Y) &= vec(BW) + vec(U) \\ &= (W' \otimes I)vec(B) + vec(U) \end{aligned}$$

  or

$$y = (W' \otimes I)\beta + u.$$

This is a linear regression model! Thus[1],

$$\hat{\beta} = [(W' \otimes I)'(W \otimes I)]^{-1}(W' \otimes I)'y.$$

[1] Recall that the OLS estimator of β in the linear regression model $y = X\beta + u$ is $\hat{\beta} = (X'X)^{-1}X'y$.

Estimating the VAR(p) model

But

$$\begin{aligned}(A \otimes B)' &= A' \otimes B' \\ (A \otimes B)(C \otimes D) &= AC \otimes BD \\ (A \otimes B)^{-1} &= A^{-1} \otimes B^{-1}.\end{aligned}$$

Thus

$$\begin{aligned}\hat{\beta} &= (WW' \otimes I)^{-1}(W \otimes I)y \\ &= [(WW')^{-1} \otimes I][W \otimes I]y \\ &= [(WW')^{-1}W \otimes I]y.\end{aligned}$$

Estimating the VAR(p) model

- This can be rewritten as

$$vec(\hat{B}) = \hat{\beta} = vec(YW'(WW')^{-1})$$

Thus

$$\hat{B} = YW'(WW')^{-1}.$$

- For the VAR(1) model,

$$\hat{\Phi} = \left(\sum r_t r_{t-1}'\right)\left(\sum r_{t-1} r_{t-1}'\right)^{-1}.$$

- We use information criteria to select the VAR order p.

Matlab

```
% y is the data matrix for 2 variables.
Spec = vgxset('n',2,'nAR',2,'Constant', true);
% k=2, ARlag = 2, constant term included
[EstSpec,EstStdErrors,LLF,W] = vgxvarx(Spec,y);
% W: residuals
vgxdisp(EstSpec,EstStdErrors);
h=10; [FY, FYCov] = vgxpred(EstSpec, h, [ ], y, W); % h = forecasting horizon
vgxplot(EstSpec, [ ], FY, FYCov);
```

Granger-causality

- Main idea: If a variable x affects a variable z, the former should help improving the predictions of the latter variables.
- To formalize the idea, let
  Ω_t : the information set containing all the relevant information in the universe available up to and including period t.
  $z_t(h \mid \Omega_t)$: the optimal (minimum MSE) h-step predictor of the process z_t at origin t, based on the information in Ω_t.
  $\Sigma_z(h \mid \Omega_t) = E(z_t(h \mid \Omega_t) - z_{t+h})^2$: the forecast MSE.

Granger-causality

- The process x_t is said to cause z_t in Granger's sense if

$$\Sigma_z(h \mid \Omega_t) < \Sigma_z\left(h \mid \Omega_t \backslash \{x_s \mid s \leq t\}\right)$$

  for at least one $h = 1, 2, \ldots$.
  $\Omega_t \backslash \{x_s \mid s \leq t\}$: all the relevant information in the universe except for the information in the past and present of the x_t process.

- In practice, we use

$$\Omega_t = \{z_s, x_s \mid s \leq t\}$$

  as an information set.

Characterization of 1-step ahead Granger-Causality

- For a stationary VAR process,

$$
\begin{aligned}
r_t &= \begin{bmatrix} z_t \\ x_t \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} \Phi_{11,1} & \Phi_{12,1} \\ \Phi_{21,1} & \Phi_{22,1} \end{bmatrix} \begin{bmatrix} z_{t-1} \\ x_{t-1} \end{bmatrix} + \ldots \\
&\quad + \begin{bmatrix} \Phi_{11,p} & \Phi_{12,p} \\ \Phi_{21,p} & \Phi_{22,p} \end{bmatrix} \begin{bmatrix} z_{t-p} \\ x_{t-p} \end{bmatrix} + \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix},
\end{aligned}
$$

  if $\Phi_{12,i} = 0$ for $i = 1, 2, \ldots,$ x_t does not help predicting z_t.
- Therefore,

$$
z_t\left(1 \mid |\{r_s \mid s \leq t\}\right) = z_t\left(1 \mid \{z_s \mid s \leq t\}\right)
$$

$$
\Leftrightarrow \Phi_{12,i} = 0 \text{ for } i = 1, \ldots, p.
$$

Granger noncausality test for stationary VAR

- Consider a stationary VAR model

$$r_t = \begin{pmatrix} z_t \\ x_{1t} \\ x_{2t} \end{pmatrix} \begin{matrix} n \\ m \\ l \end{matrix} = \sum_{i=1}^{p} \begin{bmatrix} \Phi_{11i} & \Phi_{12i} & \Phi_{13i} \\ \Phi_{21i} & \Phi_{22i} & \Phi_{23i} \\ \Phi_{31i} & \Phi_{32i} & \Phi_{33i} \end{bmatrix} \begin{bmatrix} z_{t-i} \\ x_{1(t-i)} \\ x_{2(t-i)} \end{bmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \\ a_{3t} \end{pmatrix}$$

- The null hypothesis that x_{2t} does not Granger-cause z_t at the horizon 1 can be written as

$$H_0 : \Phi_{13i} = 0 \ (i = 1, 2, ..., p).$$

Granger noncausality test for stationary VAR

- The Wald test for this null hypothesis is

$$W = vec\left(\hat{\theta}\right)'(s \otimes s_1)\left[\left(s' \otimes s_1'\right)\left[\left(v'v\right)^{-1} \otimes \hat{\Sigma}_a\right](s \otimes s_1)\right]^{-1} \times \left(s' \otimes s_1'\right) vec\left(\hat{\theta}\right)$$

where

$$s_1 = \begin{bmatrix} I_n \\ 0 \end{bmatrix}_{m+l},$$

$$s = I_p \otimes s_3 \text{ with } s_3 = \begin{bmatrix} 0 \\ I_l \end{bmatrix}_{n+m}$$

$$\hat{\theta} = \left(\sum_{t=1}^{T} r_t v_t'\right)\left(\sum_{t=1}^{T} v_t v_t'\right)^{-1}, v_t = \left[r_{t-1}', \ldots, r_{t-p}'\right]',$$

$$v = \left[v_1, \ldots, v_T\right]' \;\&\; \hat{\Sigma}_a = \frac{1}{T}\sum_{t=1}^{T}\left(r_t - \hat{\theta} v_t\right)\left(r_t - \hat{\theta} v_t\right)'.$$

Granger noncausality test for stationary VAR

- As $T \to \infty$,

$$W \xrightarrow{d} \chi^2_{nlp}.$$

Impulse response function

- A stationary VAR(p) model $r_t = \mu + \Phi_1 r_{t-1} + \cdots + \Phi_p r_{t-p} + a_t$ can be written as

$$r_t = \mu' + a_t + \Psi_1 a_{t-1} + \Psi_2 a_{t-2} + \ldots$$

where the coefficient matrices $\{\Psi_i\}$ satisfy the relation

$$(I - \Phi_1 z - \Phi_1 z^2 - \ldots - \Phi_p z^p)(I + \Psi_1 z + \Psi_2 z^2 + \ldots) = I.$$

Impulse response function

- The matrix Ψ_s has the interpretation

$$\frac{\partial r_{t+s}}{\partial a'_t} = \Psi_s.$$

  Namely, $[\Psi_s]_{ij}$ denotes the effect of a one unit increase in a_{jt} on the value of $r_{t+s,i}$.

- A plot of $[\Psi_s]_{ij}$ as a function of s is called the impulse response function. It describes the response of $r_{t+s,i}$ to a one-time impulse in r_{tj} with all other variables dated t or earlier held constant. $([\Psi_s]_{ij} = \frac{\partial r_{t+s,i}}{\partial a_{t,j}})$.

Impulse response function

- When all other variables dated t or earlier are held constant,

$$[\Psi_s]_{ij} = \frac{\partial r_{t+s,i}}{\partial a_{t,j}} = \frac{\partial r_{t+s,i}}{\partial r_{t,j}} \frac{\partial r_{t,j}}{\partial a_{t,j}} = \frac{\partial r_{t+s,i}}{\partial r_{t,j}}.$$

- But if $a_{t,j}$ and $a_{t,k}$ $(j \neq k)$ are correlated, $[\Psi_s]_{ij}$ does not capture the effect of $a_{t,j}$ on $r_{t+s,i}$ correctly since $a_{t,k}$ would also affect $r_{t+s,i}$ indirectly. That is,

$$\begin{array}{ccc} a_{t,j} & & \\ & \searrow & \\ \downarrow & & r_{t+s,i} \\ & \nearrow & \\ a_{t,k} & & \end{array}$$

Orthogonalized impulse response function

- Consider a decomposition of $\Sigma = E(a_t a_t')$

$$\Sigma = LGL'$$

where L is a lower triangular matrix with its diagonal elements being equal to one and G a diagonal matrix.

- Rewrite the original MA(∞) model such that

$$\begin{aligned} r_t &= \mu' + LL^{-1}a_t + \Psi_1 LL^{-1}a_{t-1} + \Psi_2 LL^{-1}a_{t-2} + \ldots \\ &= \mu' + \Psi_0^* b_t + \Psi_1^* b_{t-1} + \Psi_2^* b_{t-2} + \ldots \end{aligned}$$

Then,

$$E(b_t b_t') = E(L^{-1}a_t a_t L^{'-1}) = L^{-1}\Sigma_a L^{'-1} = L^{-1}LGL'L^{'-1} = G.$$

That is, the variance-covaraince matrix of b_t is diagonal. Thus, $[\Psi_s^*]_{ij}$ measure the effect of $a_{t,j}$ on $r_{t+s,i}$ correctly.

Orthogonalized impulse response function

- The plot of $[\Psi_s^*]_{ij}$ as a function of s is called the orthogonalized impulse response function.

Example

$r_t = \begin{pmatrix} \text{\# of Hyundai cars sold in the US} \\ \text{\# of Nissan, Honda, Toyota cars sold in the US} \end{pmatrix}$. The orthogonalized impulse response function $[\Psi_s^*]_{12}$ shows how the sales of Nissan, Honda, Toyota cars affect those of Hyundai cars over time.

- A major drawback of the orthogonalized impulse response function is that it depends on the ordering of the variables involved. The orthogonalized impulse response function changes as the ordering changes.

Orthogonalized impulse response function

- The reason for this is that L and Ψ change as the ordering changes.
- Consider the simple case $K = 3$ and calculate $[\Psi_s^*]_{12}$ for the original and changed orderings. Note that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \sigma_{21}\sigma_{11}^{-1} & 1 & 0 \\ \sigma_{31}\sigma_{11}^{-1} & h_{32}h_{22}^{-1} & 1 \end{bmatrix}$$

where $h_{22} = \sigma_{22} - \sigma_{21}\sigma_{11}^{-1}\sigma_{12}$, $h_{32} = \sigma_{32} - \sigma_{21}\sigma_{11}^{-1}\sigma_{13}$ and $\Sigma = [\sigma_{ij}]$ (cf. Hamilton, 1994, p.91).

Generalized impulse response function

See Pesaran, H.H. and Y. Shin (1998) "Generalized impulse response analysis in linear multivariate models," Economics Letters, 58, 17-29.

- Write

$$
\begin{aligned}
\frac{dr_{t+s,i}}{da_{t,j}} &= \frac{\partial r_{t+s,i}}{\partial a_{t,1}} \frac{\partial a_{t,1}}{\partial a_{t,j}} + \ldots + \frac{\partial r_{t+s,i}}{\partial a_{t,K}} \frac{\partial a_{t,K}}{\partial a_{t,j}} \\
&= \sum_{m=1}^{K} \frac{\partial r_{t+s,i}}{\partial a_{t,m}} \frac{\partial a_{t,m}}{\partial a_{t,j}} \\
&= \sum_{m=1}^{K} [\Psi_s]_{im} \frac{\partial a_{t,m}}{\partial a_{t,j}}.
\end{aligned}
$$

Generalized impulse response function

- Assume

$$
\begin{aligned}
a_{t,m} &= \delta_{m,j} a_{t,j} + \varepsilon_{t,m,j}, \;\; \varepsilon_{t,m,j} \sim iid(0, \sigma_\varepsilon^2), \\
& \quad \{\varepsilon_t\} \text{ and } \{a_t\} \text{ are independent.}
\end{aligned}
$$

Then, since $E(a_{t,m} a_{t,j}) = \sigma_{mj}$,

$$
E(a_{t,m} a_{t,j}) = \delta_{m,j} Var(a_{t,j})
$$

which gives

$$
\delta_{m,j} = \frac{\sigma_{mj}}{\sigma_{jj}}.
$$

Generalized impulse response function

- Since $\frac{\partial a_{t,m}}{\partial a_{t,j}} = \delta_{m,j}$, the generalized impulse response function can be written as

$$\sum_{m=1}^{K} [\Psi_s]_{im} \frac{\sigma_{mj}}{\sigma_{jj}}.$$

The parameter $\frac{\sigma_{mj}}{\sigma_{jj}}$ can be estimated by using the sample variance-covariance matrix from the VAR analysis.

- Some authors prefer using

$$\frac{\partial r_{t+s,i}}{\partial (a_{t,j}/\sqrt{\sigma_{jj}})}.$$

This denotes the change in $r_{t+s,i}$ per one standard deviation change in $a_{t,j}$.

Generalized impulse response function

- The scaled generalized impulse response function is written as

$$\sum_{m=1}^{K} [\Psi_s]_{im} \frac{\sigma_{mj}}{\sqrt{\sigma_{jj}}}.$$

Forecast error variance decomposition

- Suppose that $\{r_t\}$ is a $K \times 1$ vector linear process written as

$$\begin{aligned} r_t &= \mu + \sum_{i=0}^{\infty} \Psi_i P P^{-1} a_{t-i} \\ &= \mu + \sum_{i=0}^{\infty} \Theta_i w_{t-i}, \end{aligned}$$

where $\Theta_i = \Phi_i P$, $w_t = P^{-1} a_t$ and $E(w_t w_t') = I$ for all t.

Forecast error variance decomposition

- The optimal h-step forecast is

$$r_t(h) = E(r_{t+h} \mid r_t, r_{t-1}, \ldots) = \mu + \sum_{i=h}^{\infty} \Theta_i w_{t+h-i}.$$

The forecast error is

$$r_{t+h} - r_t(h) = \sum_{i=0}^{h-1} \Theta_i w_{t+h-i}.$$

- The mn-th element of Θ_i is denoted as $\theta_{mn,i}$, and the h-step forecast error of the j-th component of r_t is

$$\begin{aligned} r_{j,t+h} - r_{j,t}(h) &= \sum_{i=0}^{h-1} \left(\theta_{j1,i} w_{1,t+h-i} + \ldots + \theta_{jK,i} w_{K,t+h-i}\right) \\ &= \sum_{k=1}^{K} \left(\theta_{jk,0} w_{k,t+h} + \ldots + \theta_{jk,h-1} w_{k,t+1}\right). \end{aligned}$$

Forecast error variance decomposition

- The MSE of the forecast error is

$$E(r_{j,t+h} - r_{j,t}(h))^2 = \sum_{k=1}^{K} \left(\theta_{jk,0}^2 + ... + \theta_{jk,h-1}^2\right).$$

  Here $\theta_{jk,0}^2 + ... + \theta_{jk,h-1}^2$ is the contribution of the k-the variable to the MSE.

- The quantity $\omega_{jk,h} = \left(\theta_{jk,0}^2 + ... + \theta_{jk,h-1}^2\right) / \sum_{k=1}^{K} \left(\theta_{jk,0}^2 + ... + \theta_{jk,h-1}^2\right)$ is the proportion of the h-step forecast error variance of variable j accounted for by the k-th variable. The quantities $\{\omega_{jk,h}\}$ constitute the forecast error variance decomposition.