Homework 4

Mathematical Statistics (Fall, 2018)
Total points: 10
Due date: 11. 5 (Tu)

1. Let $X_1,\ X_2,\ X_3$ be iid $N(\mu, \sigma^2)$ and define
$$Y_1 = X_1 + \delta X_3$$
and
$$Y_2 = X_2 + \delta X_3.$$
   (a) Find the means and variances of Y_1 and Y_2 and their correlation coefficients.
   (b) Find the joint mgf of Y_1 and Y_2.

2. Let X and Y have a bivariate normal distribution. Show that $X+Y$ and $X-Y$ are independent if and only if $VarX = VarY$.

3. Prove that $ES \le \sigma$. [Hint: use Jensen's inequality.]

4. Let $X_i \sim iid\ N(\mu, \sigma^2)$. Find the mean and variance of $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$. [Hint: use the distribution of $(n-1)S^2/\sigma^2$.]

5. Let X be a random variable with a Student's t-distribution with p degrees of freedom. Derive the mean and variance of X.

6. Let $X_1,\ X_2, ...$ be uncorrelated with $EX_i = \mu_i$ and $Var(X_i)/i \to 0$ as $i \to 0$. Let $S_n = \sum_{i=1}^n X_i$ and $\upsilon_n = ES_n/n$. Show that $S_n/n - \upsilon_n \xrightarrow{p} 0$ as $n \to \infty$.

7. Suppose $EX_n = 0$ and $EX_nX_m \le \gamma(n-m)$ for $m \le n$ with $\gamma(k) \to 0$ as $k \to \infty$. Show that $\sum_{i=1}^n X_i/n \xrightarrow{p} 0$ as $n \to \infty$.

8. Show that if $\sqrt{n}(Y_n - \mu) \xrightarrow{d} N(0, \sigma^2)$, then $Y_n \xrightarrow{p} \mu$.

9. Let $X_i,\ i = 1, 2, ...,$ be independent $Bernoulli(p)$ random variables and let $Y_n = \sum_{i=1}^n X_i/n$.
   (a) Show that $\sqrt{n}(Y_n - p) \xrightarrow{d} N(0, p(1-p))$.
   (b) Show that for $p \ne 1/2$,
$$\sqrt{n}(Y_n(1-Y_n) - p(1-p)) \xrightarrow{d} N(0, (1-2p)^2 p(1-p)).$$
   (c) Show that for $p = 1/2$, $n[Y_n(1-Y_n) - 1/4] \xrightarrow{d} -\chi_1^2/4$.

10. Show that Lyapounov's condition
$$\lim \sum_{i=1}^{r_n} \frac{1}{s_n^{2+\delta}} E\left|X_{ni}\right|^{2+\delta} = 0 \text{ for } \delta > 0$$
implies Lindeberg's condition.