Homework 2

Mathematical Statistics (Fall, 2018)
Total points: 10
Due date: Oct. 1 (W)

1. Let X be a continuous, nonnegative random variable with pdf f. Show that

$$EX = \int_0^\infty (1 - F_X(x))dx.$$

2. Let X be a random variable such that $E(X-b)^2$ exists for all real b. Show that $E(X-b)^2$ is a minimum when $b = EX$.

3. Show that if X is a continuous random variable, then

$$\min E\left|X - a\right| = E\left|X - m\right|,$$

where m is the median of X.

4. Let X have the pdf

$$f(x) = \frac{1+x}{2}, \ -1 < x < 1.$$

Find the pdf of $Y = X^2$.

5. Let X be a random variable having the pdf $\frac{2x}{\pi^2} I_{(0,\pi)}(x)$. Derive the pdf of $Y = \sin X$.

6. Let $X_1, ..., X_k$ be independent random variables and $Y = X_1 + ... + X_k$. Prove the following.

   (a) If $X_i \sim Binomial(n_i, p)$, $Y \sim Binomial(n_1 + ... + n_k, p)$.

   (b) If X_i has the Cauchy distribution, Y/k has the same distribution as X_1.

7. Let X has a standard normal distribution. Define a discrete random variable Y by

$$P(Y = \sqrt{3}) = P(Y = -\sqrt{3}) = \frac{1}{6}, \ P(Y = 0) = \frac{2}{3}.$$

Show that $EX^r = EY^r$ for $r = 1, 2, 3, 4, 5$.

8. Let μ_n denote the nth central moment of a random variable X. Two quantities of interest, in addition to the mean and variance, are

$$\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} \text{ and } \alpha_4 = \frac{\mu_4}{\mu_2^2}.$$

The value α_3 is called the skewness and α_4 is called the kurtosis. The skewness measures the lack of symmetry in the pdf. The kurtosis, although harder to interpret, measures the peakedness or flatness of the pdf.

(a) Show that if a pdf is symmetric about a point a, then $\alpha_3 = 0$.

(b) Calculate α_3 for $f(x) = e^{-x},\ x \geq 0$, a pdf that is skewed to the right.

(c) Calculate α_4 for each of the following pdfs and comment on the peakedness of each.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},\ -\infty < x < \infty$$

$$f(x) = \frac{1}{2},\ -1 < x < 1.$$

$$f(x) = \frac{1}{2} e^{-|x|}.$$

9. Find the moment generating functions of the following pdfs

(a) $f(x) = \frac{1}{c},\ 0 < x < c.$

(b) $f(x) = \frac{2x}{c^2},\ 0 < x < c.$

10. Let X have the distribution function

$$\begin{aligned} F(x) &= 0,\ x < 0 \\ &= \frac{x+1}{4},\ 0 \leq x < 1 \\ &= 1,\ 1 \leq x. \end{aligned}$$

Find EX and $VarX$.