Homework 1

Mathematical Statistics (Fall, 2018)
Total points:
Due date: 9. 13 (Th)

1. Let S be a sample space.

   (a) Show that the collection $\mathcal{B} = \{0, S\}$ is a sigma algebra.

   (b) Let $\mathcal{B} = \{\text{all subsets of } S, \text{ including } S \text{ itself}\}$. Show that $\mathcal{B}$ is a sigma algebra.

   (c) Show that the intersection of two sigma algebras is a sigma algebra.

2. Suppose that a sample space S has n elements. Prove that the number of subsets that can be formed from the elements of S is 2^n.

3. If n balls are placed at random into n cells, find the probability that exactly one cell remains empty.

4. Two players, A and B, alternately and independently flip a coin and the first player to obtain a head wins. Assume player A flips first.

   (a) If the coin is fair, what is the probability that A wins?

   (b) Suppose that $P(head) = p$, not necessarily $1/2$. What is the probability that A wins?

   (c) Show that for all p, $0 < p < 1$, $P(\text{A } wins) > 1/2$.

5. An employer is about to hire one new employee from a group of N candidates, whose future potential can be rated on a scale from 1 to N. The employer proceeds according to the following rules:
   (a) Each candidate is seen in succession (in random order) and a decision is made whether to hire the candidate.
   (b) Having rejected $m-1$ candidates ($m > 1$), the employer can hire the mth candidate only if the mth candidate is better than the previous $m-1$.
   Suppose a candidate is hired on the ith trial. What is the probability that the best candidate was hired?

6. Prove each of the following statements. (Assume that any conditioning event has positive probability.)

   (a) If $P(B) = 1$, then $P(A \mid B) = P(A)$ for any A.

   (b) If $A \subset B$, then $P(B \mid A) = 1$ and $P(A \mid B) = P(A)/P(B)$.

   (c) If A and B are mutually exclusive, then

$$P(A \mid A \cup B) = \frac{P(A)}{P(A) + P(B)}.$$

(d) $P(A \cap B \cap C) = P(A \mid B \cap C)P(B \mid C)P(C)$.

7. A pair of events A and B cannot be simultaneously mutually exclusive and independent. Prove that if $P(A) > 0$ and $P(B) > 0$, then:

   (a) If A and B are mutually exclusive, they cannot be independent.

   (b) If A and B are independent, they cannot be mutually exclusive.

8. Prove that the following functions are cdfs.

   (a) $(1 + e^{-x})^{-1}$, $x \in R$

   (b) $1 - e^{-x}$, $x \in (0, \infty)$

9. Let X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. For a fixed number x_0, define the function

$$g(x) = \{ \begin{array}{cc} f(x)/(1 - F(x_0)), & x \geq x_0 \\ 0 & x < x_0 \end{array} .$$

   Prove that $g(x)$ is a pdf. (Assume that $F(x_0) < 1$.)

10. A certain river floods every year. Suppose that the low-water mark is set at 1 and the high-water mark Y has distribution function

$$F_Y(y) = P(Y \leq y) = 1 - \frac{1}{y^2}, \quad 1 \leq y < \infty.$$

   (a) Verify that $F_Y(y))$ is a cdf.

   (b) Find the pdf of Y.

   (c) If the low-water mark is reset at 0 and we use a unit of measurement that is 1/10 of that given previously, the high-water mark becomes $Z = 10(Y - 1)$. Find $F_Z(z)$.