Financial Econometrics

Chapter 1: Basic Concepts

In Choi

Sogang University

Examples of financial time series

- Stock prices, stock returns, interest rates, exchange rates, etc.
- These series are volatile and difficult to forecast.
- Understanding properties of these series is important to bankers, investors and policy-makers.

Let P_t be the price of an asset at time t.
One-period simple net return R_t is defined by the relation

$$1 + R_t = \frac{P_t}{P_{t-1}} \text{ or } P_t = P_{t-1}(1 + R_t),$$

which gives

$$R_t = \frac{P_t}{P_{t-1}} - 1 = \frac{P_t - P_{t-1}}{P_{t-1}}.$$

Example

Let $P_{March} = 102$ and $P_{April} = 103$. Then,

$$R_{April} = \frac{103 - 102}{102} = 0.0098.$$

Holding the asset for k periods between dates $t-k$ and t gives a k-period simple gross return

$$\begin{aligned} 1+R_t[k] &= \frac{P_t}{P_{t-k}} = \frac{P_t}{P_{t-1}} \times \frac{P_{t-1}}{P_{t-2}} \times \cdots \times \frac{P_{t-k+1}}{P_{t-k}} \\ &= (1+R_t)(1+R_{t-1})\cdots(1+R_{t-k+1}) \\ &= \Pi_{j=0}^{k-1}(1+R_{t-j}). \end{aligned}$$

The k-period simple net return is

$$R_t[k] = \frac{P_t - P_{t-k}}{P_{t-k}} = \Pi_{j=0}^{k-1}(1+R_{t-j}) - 1.$$

Example

Suppose the yearly closing prices of a stock are:

Year	1	2	3	4	5
Price	37.84	38.49	37.12	37.60	36.30

1. What is the simple net return from year 1 to year 2?
Ans: $R_2 = \frac{38.49-37.84}{37.84} = 1.7178 \times 10^{-2}$.
2. What is the simple net return from year 1 to year 5?
Ans: $R_5(4) = \frac{36.30-37.84}{37.84} = -4.0698 \times 10^{-2}$.

If an asset is held for k years, the annualized average return is defined by the relation

$$(1 + Annualized\{R_t[k]\})^k = \Pi_{j=0}^{k-1}(1 + R_{t-j}).$$

This gives

$$Annualized\{R_t[k]\} = \left[\Pi_{j=0}^{k-1}(1 + R_{t-j})\right]^{1/k} - 1.$$

$\left[\Pi_{j=0}^{k-1}(1 + R_{t-j})\right]^{1/k}$ is the geometric mean of $\{1 + R_{t-j}\}_{j=0,\dots,k-1}$. This can be calculated by

$$\begin{aligned} Annualized\{R_t[k]\} &= \exp\left[\ln\left[\Pi_{j=0}^{k-1}(1 + R_{t-j})\right]^{1/k}\right] - 1 \\ &= \exp\left[\frac{1}{k}\sum_{j=0}^{k-1}\ln(1 + R_{t-j})\right] - 1. \end{aligned}$$

Asset returns

Multiperiod simple return

Example

The annual simple net returns are

Year	2	3	4	5
Return	1.7178×10^{-2}	-3.5594×10^{-2}	1.2931×10^{-2}	$-3.4574 \times 10^{-}$

Thus, the annualized average return for the 4 year period is

$$\begin{aligned} & \exp\{\frac{1}{4}[(\ln(1+1.7178\times 10^{-2}) + \ln(1-3.5594\times 10^{-2}) \\ & + \ln(1+1.2931\times 10^{-2}) + \ln(1-3.4574\times 10^{-2})]\} - 1 \\ = \; & -1.0333\times 10^{-2}. \end{aligned}$$

This is different from the arithmetic average of the net returns from the 4 year period

$$(1.7178 - 3.5594 + 1.2931 - 3.4574) \times 10^{-2}/4 = -1.0015 \times 10^{-2}.$$

Using the annualized net return for the 4 year period, the simple net return from year 1 to year 5 is calculated by

$$(1 - 0.01033)^4 - 1 = -4.0684 \times 10^{-2}.$$

If we use the average, the return becomes

$$(1 - 0.010015)^4 - 1 = -3.9462 \times 10^{-2}$$

which is different from -4.0698×10^{-2} calculated in the previous example. These calculation illustrate that using the geometric mean is better than using the arithmetic mean.

Suppose that the interest rate of a bank deposit is 2% per annum. If the bank pays interest once a year, the net value of the \$1 deposit after a year will be

$$\$(1+0.02) = \$1.02.$$

If the bank pay the interest twice a year, it will be

$$\$(1+0.02/2)^2 = \$1.0201.$$

In general, if the annual interest rate is r and the bank pays interest m times a year, the net value of the \$C deposit after a year will be

$$C(1+\frac{r}{m})^m.$$

In continuous compounding m is infinite and

$$\lim_{m\to\infty}(1+\frac{r}{m})^m = e^r.$$

Thus, when compounded continuously, the net value after a year will be

$$Ce^r.$$

In addition, the net value after n years will be

$$A = Ce^{rn}.$$

This equation gives

$$C = Ae^{-rn},$$

which is referred as the present value of an asset that is worth A dollars n years later.

Suppose that

$$e^{r_t} = 1 + R_t.$$

Then, r_t is the interest rate that provides the gross return $1 + R_t$ with continuous compounding. Thus, we call r_t the continuously compounded return. It is also called the log return because it is written as

$$r_t = \ln(1 + R_t) = \ln\left(\frac{P_t}{P_{t-1}}\right) = p_t - p_{t-1}.$$

Note that

$$r_t \simeq R_t \quad (\ln(1 + x) \simeq x).$$

For the continuously compounded multiperiod return,

$$\begin{aligned} r_t[k] &= \ln(1 + R_t[k]) \\ &= \ln(1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1}) \\ &= \ln(1 + R_t) + \ln(1 + R_{t-1}) + \cdots + \ln(1 + R_{t-k+1}) \\ &= r_t + r_{t-1} + \ldots + r_{t-k+1}. \end{aligned}$$

Example

Suppose the yearly closing prices of a stock are:

Year	1	2	3	4	5
Price	37.84	38.49	37.12	37.60	36.30

1. What is the log return from year 1 to year 2? Ans: $\ln(38.49) - \ln(37.84) = 1.7032 \times 10^{-2}$. (Note that $R_2 = \frac{38.49-37.84}{37.84} = 1.7178 \times 10^{-2}$.)
2. What is the log return from year 1 to year 5? Ans: $\ln(36.30) - \ln(37.84) = -4.1549 \times 10^{-2}$. (note that $R_5[4] = \frac{36.30-37.84}{37.84} = -4.0698 \times 10^{-2}$.)

There are N assets with simple returns $R_{1t}, \ldots, R_{Nt}$. Let p be a portfolio that places weights w_i on asset i. Then, the simple net return of p at time t is

$$R_{p,t} = \sum_{i=1}^{N} w_i R_{it}.$$

Example

An investor holds stocks of IBM, Microsoft and Citi-Group. Assume that her capital allocation is 30%, 30% and 40%. The monthly simple net returns are 1.42, 4.26 and 2.55, respectively. Then the mean simple net return of the portfolio is $0.3 \times 1.42 + 0.3 \times 4.26 + 0.4 \times 2.55 = 2.72$.

When dividend D_t is paid out between dates $t-1$ and t, the simple net return is defined as

$$R_t = \frac{P_t + D_t}{P_{t-1}} - 1 = \frac{P_t + D_t - P_{t-1}}{P_{t-1}}.$$

The continuously compounded return becomes

$$r_t = \ln(P_t + D_t) - \ln(P_{t-1}).$$

Let R_{ot} be the return for a risk-free asset and $r_{ot} = \ln(1 + R_{ot})$. The simple excess return and log excess return are defined as

$$Z_t = R_t - R_{ot}, \; z_t = r_t - r_{ot}.$$

Let P_t^g be a price index for goods and services at year t. With the \$C deposit and the nominal deposit rate R at year 1, the real deposit amount a year later will be

$$C_2^r = \frac{C(1+R)}{P_2^g}.$$

Since the real deposit at year 1 is $C_i^r = \frac{C}{P_1^g}$, the real net return of the deposit, R^r, is defined by the equation

$$(1 + R^r) = \frac{C_2^r}{C_1^r} = (1 + R)\frac{1}{P_2^g \,/\, P_1^g} = \frac{1 + R}{1 + \Pi}$$

where $1 + \Pi = P_2^g \,/\, P_1^g$. Here Π is the inflation rate. The real return is written as

$$R^r = \frac{1 + R}{1 + \Pi} - 1 = \frac{R - \Pi}{1 + \Pi} \simeq R - \Pi.$$

In terms of continuously compounded returns,

$$r^r = \ln(1+R^r) = \ln(1+R) - \ln(1+\Pi) = r - \pi$$

where r and π are continuously compounded nominal return and inflation rate, respectively.

For the continuously compounded multiperiod real return,

$$\begin{aligned} r_t^r[k] &= \ln(1+R_t^r[k]) \\ &= \ln(1+R_t^r)(1+R_{t-1}^r)\cdots(1+R_{t-k+1}^r) \\ &= \ln(1+R_t^r) + \ln(1+R_{t-1}^r) + \cdots + \ln(1+R_{t-k+1}^r) \\ &= r_t^r + r_{t-1}^r + \ldots + r_{t-k+1}^r. \end{aligned}$$

If the deposit is held for k years, the annualized average real return is defined by the relation

$$(1 + Annualized\{R_t^r[k]\})^k = \Pi_{j=0}^{k-1}(1 + R_{t-j}^r).$$

This gives

$$Annualized\{R_t^r[k]\} = \left[\Pi_{j=0}^{k-1}(1 + R_{t-j}^r)\right]^{1/k} - 1.$$

This can be calculated by

$$\begin{aligned} Annualized\{R_t^r[k]\} &= \exp\left[\ln\left[\Pi_{j=0}^{k-1}(1 + R_{t-j}^r)\right]^{1/k}\right] - 1 \\ &= \exp\left[\frac{1}{k}\sum_{j=0}^{k-1}\ln(1 + R_{t-j}^r)\right] - 1. \end{aligned}$$

Discounted present value

The discounted present value of the cash receipts $C_1, ..., C_n$ that carries no default risk is given by

$$DPV = \sum_{i=1}^{n} \frac{C_i}{(1+R)^i}$$

where R is the interest rate that does not change over the period.

If the future payments are constant each year and the first payment is at the end of the first year, these payment scheme is called ordinary annuity. The DPV of these payments is

$$DPV = \sum_{i=1}^{n} \frac{C}{(1+R)^i}.$$

When n is large, $DPV \simeq C/R$.

The fundamental value of a stock, V_t, that will pay dividends $D_{t+1}, D_{t+2}, ...$ is the expected DPV of these dividend stream.

$$V_t = E_t[\frac{D_{t+1}}{1+R_t} + \frac{D_{t+2}}{(1+R_t)(1+R_{t+1})} + ...]$$

where R_i is the discount rate at period t.

The l-th moment of a continuous random variables is defined by

$$m_l' = E(X^l) = \int_{-\infty}^{\infty} x^l f(x) dx$$

where $f(x)$ is the probability density function of X.

- First moment (m_1'): mean or expectation of X, μ_X.
- Mean is related to long-term return.

The l-th *central* moment of a continuous random variables is defined by

$$m_l = E(X - \mu_X)^l = \int_{-\infty}^{\infty} (x - \mu_X)^l f(x) dx.$$

- Second central moment: variance of X, σ_X^2.
- Variance measures risk.
- Skewness: $S_X = E[\frac{(X-\mu_x)^3}{\sigma_x^3}]$, measures the extent of asymmetry.

- Kurtosis: $K_X = E[\frac{(X-\mu_x)^4}{\sigma_x^4}]$, measures the extent of tail thickness.
- $K_X - 3$: excess kurtosis ($K_X = 3$ for a normal distribution).
- Positive excess kurtosis: heavy tails, extreme values
- Negative excess kurtosis: short tails
- Kurtosis is related to volatility forecasting, efficiency in estimation and tests, etc.

The sample mean is

$$\hat{\mu}_X = \frac{1}{T}\sum_{t=1}^{T} X_t.$$

The sample variance is

$$\hat{\sigma}_X^2 = \frac{1}{T-1}\sum_{t=1}^{T}(X_t - \hat{\mu}_X)^2$$

The sample skewness is

$$\hat{S}_X = \frac{1}{(T-1)\hat{\sigma}_X^3} \sum_{t=1}^{T} (X_t - \hat{\mu}_X)^3$$

The sample kurtosis is

$$\hat{K}_X = \frac{1}{(T-1)\hat{\sigma}_X^4} \sum_{t=1}^{T} (X_t - \hat{\mu}_X)^4$$

When $X_t \stackrel{d}{=} N(\mu, \sigma^2)$,

$$\hat{S}_X \simeq N(0, \frac{6}{T}) \text{ and } \hat{K}_X - 3 \simeq N(0, \frac{24}{T}).$$

- Test for symmetry:

$$S = \frac{\hat{S}_X}{\sqrt{\frac{6}{T}}} \simeq N(0,1),$$

  if normality holds.

Decision rules: Reject $Ho : symmetry$ at the 5% level if $|S| > 1.96$.

- Test for either tail thickness or tail shortness:

$$K = \frac{\hat{K}_X - 3}{\sqrt{\frac{24}{T}}} \simeq N(0,1)$$

  if normality holds.

Decision rule: (1) Reject the null of normal tails against the alternative of thick tails at the 5% level if $K >$1.65.
(2) Reject the null of normal tails against the alternative of thin tails at the 5% level if $K <$-1.65.

- A test for normality (Jarque-Bera test):

$$JB = S^2 + K^2 \simeq \chi^2(2),$$

  if normality holds.

Decision rule: Reject the null hypothesis of normality at the 5% level if $JB > 5.99$.

- If $r \sim N(\mu, \sigma^2)$, $R = \exp(r)$ is lognormal with mean and variance

$$E(R) = \exp(\mu + \frac{\sigma^2}{2}) \text{ and } Var(R) = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1].$$

- Conversely, if R is lognormal with mean μ_R and variance σ_R^2,

$$r = \ln(R) \sim N\left(\ln\left(\frac{\mu_R}{\sqrt{1 + \frac{\sigma_R^2}{\mu_R^2}}}\right), \ln\left(1 + \frac{\sigma_R^2}{\mu_R^2}\right)\right).$$

- High excess kurtosis is observed.
- Means are positive.
- Skewness is observed.

Market efficiency

A market in which prices fully reflect available information is called efficient.

- Weak-form efficiency: The information set includes on the history of prices or returns themselves.
- Semistrong-form efficiency: The information set includes all publicly available information.
- Strong-form efficiency: The information set includes all public and private information.

Exercises

1. Using daily, simple returns of the KOSPI200 index during the period 2010-2012 (on the basis of closing prices), perform the following.
    1. Test the null of symmetry at the 5% level.
    2. Test the null of normal tails at the 5% level.
    3. Test the null of normality at the 5% level.
    4. Plot the sample autocorrelation function. Do they indicate serial correlation?
    5. Test the null of no serial correlation using the Ljung-Box test at the 5% level.
2. Using weekly and monthly price data of the the KOSPI200 index, perform the same exercises as in Problem 1. Are there any noticeable differences?