Introduction to Econometrics

Chapter 20: Dynamic Panels

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Linear Models for Panel Data

Useful references:

- Badi Baltagi (2008) *Econometric Analysis of Panel Data*, 4th Edition, John Wiley and Sons.
- Cheng Hsiao (2003) *Analysis of Panel Data*, 2nd Edition, Cambridge University Press.
- Cameron, A.C., and P.K. Trivedi (2005). Microeconometrics: Methods and Applications. Cambridge University Press: New York.
- Wooldridge, J. (2002). *Econometric Analysis of Cross Section and Panel Data*, The MIT Press.

- Consider the model

$$y_{it} = \alpha + \gamma y_{i,t-1} + \mu_i + v_{it}, \ (t = 1, ..., T; i = 1, ..., N).$$

- Assume
  1. $|\gamma| < 1$.
  2. y_{i0} are observable.
  3. $v_{it} \sim iid(0, \sigma^2)$ for all i.
  4. $\{v_{1t}\}, ..., \{v_{Nt}\}$ are independent.
  5. $E(v_{it}\mu_i) = 0$ for all i and t.

- The LSDV estimator of γ is

$$
\begin{aligned}
\hat{\gamma} &= \frac{\sum_{i=1}^{N}\sum_{t=1}^{T}(y_{it}-\bar{y}_i)(y_{i,t-1}-\bar{y}_{i,-1})}{\sum_{i=1}^{N}\sum_{t=1}^{T}(y_{i,t-1}-\bar{y}_{i,-1})^2} \qquad (1) \\
&= \gamma + \frac{\sum_{i=1}^{N}\sum_{t=1}^{T}(y_{i,t-1}-\bar{y}_{i,-1})(v_{i,t}-\bar{v}_i)/NT}{\sum_{i=1}^{N}\sum_{t=1}^{T}(y_{i,t-1}-\bar{y}_{i,-1})^2/NT}.
\end{aligned}
$$

Continuous substitution gives

$$y_{it} = v_{it} + \gamma v_{i,t-1} + \ldots + \gamma^{t-1} v_{i1} + \frac{1-\gamma^t}{1-\gamma}(\alpha + \mu_i) + \gamma^t y_{i0}.$$

Summing $y_{i,t-1}$ over t, we have

$$\begin{aligned} \sum_{t=1}^{T} y_{i,t-1} &= \frac{1-\gamma^T}{1-\gamma} y_{i0} + \frac{(T-1) - T\gamma + \gamma^T}{(1-\gamma)^2}(\alpha + \mu_i) \\ &\quad + \frac{1-\gamma^{T-1}}{1-\gamma} v_{i1} + \frac{1-\gamma^{T-2}}{1-\gamma} v_{i2} + \ldots + v_{i,T-1}. \end{aligned} \tag{2}$$

To analyze the probability limit of the numerator of estimator (1), consider the relations

$$
\begin{aligned}
& p\lim_{N\to\infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (y_{i,t-1} - \bar{y}_{i,-1})(v_{i,t} - \bar{v}_i) \\
= \;& p\lim_{N\to\infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (y_{i,t-1} - \bar{y}_{i,-1}) v_{i,t} \\
= \;& -p\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \bar{y}_{i,-1} \bar{v}_i,
\end{aligned}
$$

where the second equality follows since $p\lim_{N\to\infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} y_{i,t-1} v_{i,t} = 0$.

But using (2) and the given assumptions, we find

$$-p\lim_{N\to\infty}\frac{1}{NT}\sum_{i=1}^{N}\bar{y}_{i,-1}\bar{v}_i = -\frac{\sigma^2}{T^2}\frac{(T-1)-T\gamma+\gamma^T}{(1-\gamma)^2}, \tag{3}$$

which is the probability limit of the numerator of estimator (1). Similarly, for the denominator of estimator (1), we obtain

$$\begin{aligned} & p\lim_{N\to\infty}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{i,t-1}-\bar{y}_{i,-1}\right)^2/NT \\ = \; & \frac{\sigma^2}{1-\gamma^2}\left\{1-\frac{1}{T}-\frac{2\gamma}{(1-\gamma)^2}\frac{(T-1)-T\gamma+\gamma^T}{T^2}\right\}. \end{aligned} \tag{4}$$

Remarks (i) Relations (3) and (4) show that $\hat{\gamma}$ is inconsistent for fixed T.
(ii) This is caused by having to eliminate the unknown individual effects μ_i from each observation, which creates correlation between $y_{i,t-1} - \bar{y}_{i,-1}$ and $v_{i,t} - \bar{v}_i$.
(iii) The inconsistency of the LSDV estimator holds whether μ_i are random or fixed.
(iv) The asymptotic bias will die out as T increases.

- Consider the random effects model

$$y_{it} = \gamma y_{i,t-1} + \mu_i + v_{it}, (t = 1, ..., T; i = 1, ..., N), \tag{5}$$

where μ_i is a random variable.

- The OLS estimator of γ is

$$\hat{\gamma} = \frac{\sum_{i=1}^{N}\sum_{t=1}^{T} y_{it} y_{i,t-1}}{\sum_{i=1}^{N}\sum_{t=1}^{T} y_{i,t-1}^2} = \gamma + \frac{\sum_{i=1}^{N}\sum_{t=1}^{T} y_{it}(\mu_i + v_{it})}{\sum_{i=1}^{N}\sum_{t=1}^{T} y_{i,t-1}^2}.$$

Using the same methods as in the previous subsection, we have

$$p \lim_{N\to\infty} \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} y_{it}(\mu_i + v_{it})$$
$$= \frac{1}{T}\frac{1-\gamma^T}{1-\gamma} Cov(y_{i0}, \mu_i) + \frac{1}{T}\frac{\sigma^2}{(1-\gamma)^2}\left[(T-1) - T\gamma + \gamma^T\right]$$

and

$$\begin{aligned} p\lim_{N\to\infty}\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}y_{i,t-1}^{2} &= \frac{1-\gamma^{2T}}{T(1-\gamma^{2})}\frac{\sum_{i=1}^{N}y_{i0}^{2}}{N} \\ &+\frac{\sigma^{2}}{(1-\gamma)^{2}}\frac{1}{T}\left(T-2\frac{1-\gamma^{T}}{1-\gamma}+\frac{1-\gamma^{2T}}{1-\gamma^{2}}\right) \\ &+\frac{2}{T(1-\gamma)}\left(\frac{1-\gamma^{T}}{1-\gamma}-\frac{1-\gamma^{2T}}{1-\gamma^{2}}\right)Cov(\mu_{i},y_{i0} \\ &+\frac{\sigma^{2}}{T(1-\gamma^{2})^{2}}\left[(T-1)-T\gamma^{2}+\gamma^{2T}\right]. \end{aligned}$$

Thus, the OLS estimator is not consistent.

- Taking the difference of model (5), we obtain

$$y_{it} - y_{i,t-1} = \gamma \left(y_{i,t-1} - y_{i,t-2} \right) + v_{it} - v_{i,t-1}.$$

  Since $y_{i,t-2} - y_{i,t-3}$ is uncorrelated with $v_{it} - v_{i,t-1}$ and correlated with $y_{i,t-1} - y_{i,t-2}$, it can be used as an instrument (cf. Anderson and Hsiao, 1981, JASA).

- For $t = 3$, we have

$$y_{i3} - y_{i2} = \gamma \left(y_{i,2} - y_{i,1} \right) + v_{i3} - v_{i,2}.$$

  Thus, y_{i1} is a valid instrument.

- For $t = 4$, we have

$$y_{i4} - y_{i3} = \gamma \left(y_{i3} - y_{i2} \right) + v_{i4} - v_{i3}$$

  In this case, y_{i1} and y_{i2} are valid instruments.
- For period T, the set of instruments becomes $\left(y_{i1}, y_{i2}, ..., y_{i.T-2} \right)$.

- Letting $\Delta v_i = (v_{i3} - v_{i2}, ..., v_{iT} - v_{i,T-1})'$, we find

$$E\Delta v_i \Delta v_i' = \sigma_v^2 G,$$

where $G = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$

- Define

$$W_i = \begin{pmatrix} [y_{i1}] & & & 0 \\ & [y_{i1}, y_{i2}] & & \\ & & \ddots & \\ 0 & & & [y_{i1}, y_{i2}, ..., y_{i.T-2}] \end{pmatrix}$$

and $W = [W_1', ..., W_N']'$. Premultiplying the differenced equation by W, we get

$$W'\Delta y = W'\Delta y_{-1}\gamma + W'\Delta v.$$

Performing GLS on this model, Arellano and Bond (1991, RES) obtains

$$\begin{aligned} \hat{\gamma} &= \left[(\Delta y_{-1})' W(W'(I_N \otimes G)W)^{-1} W' (\Delta y_{-1})\right]^{-1} \\ &\quad \times \left[(\Delta y_{-1})' W(W'(I_N \otimes G)W)^{-1} W' (\Delta y)\right]. \end{aligned}$$

- The optimal GMM estimator is

$$\tilde{\gamma} = \left[(\Delta y_{-1})' W (\sum_{i=1}^{N} W_i' (\Delta v_i) (\Delta v_i)' W)^{-1} W' (\Delta y_{-1}) \right]^{-1} \times \left[(\Delta y_{-1})' W (\sum_{i=1}^{N} W_i' (\Delta v_i) (\Delta v_i)' W)^{-1} W' (\Delta y) \right].$$

To make this estimator operational, replace Δv_i with $\Delta \hat{v}_i$ obtained from $\hat{\gamma}$.

- See also Arellano and Bover (1995, JoE), Blundell and Bond (1998, JoE) for related research.