Econometrics

Chapter 18: Cointegration

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- A series with no deterministic component which has a stationary, invertible, ARMA representation after differencing d times, is said to be integrated of order d, denoted $x_t \sim I(d)$.

Example If

$$y_t = y_{t-1} + u_t, u_t \sim WN(0, \sigma^2),$$

$y_t \sim I(1)$.

Example If

$$y_t = u_t + 0.5u_{t-1}, \; u_t \sim WN(0, \sigma^2),$$

$y_t \sim I(0)$.

- Properties of $I(1)$ series
  - (i) Growing variance. ($y_t \sim I(1)$. Then $var(y_t) \longrightarrow \infty$ as $t \longrightarrow \infty$.)
  - (ii) An innovation has a permanent effect on the value of y_t. ($y_t = \sum_{-\infty}^{t} u_i$.)
  - (iii) The expected time between crossings of $x = 0$ is infinite.
  - (iv) The theoretical autocorrelation, $\rho_k \longrightarrow 1$ for all k as $t \longrightarrow \infty$.

Cointegration

- If $x_t \sim I(1)$ and $y_t \sim I(1)$, it is generally true that $z_t = y_t - ax_t \sim I(1)$. When $z_t \sim I(0)$, we say that x_t and y_t are cointegrated.

Example (i) x_t : dividend, y_t : stock price
(ii) x_t : income, y_t : consumption
(iii) x_t : interest rate, y_t : money

Cointegration

- The coefficient a represents the long-run equilibrium relationship among variables.
- Consider the regression model

$$y_t = \mu + \alpha x_t + u_t, \ (t = 1, ..., T)$$

  where $x_t \sim I(1)$ and $u_t \sim I(0)$.
- Even if Δx_t and u_t are correlated,

$$\hat{\alpha} \xrightarrow{p} \alpha.$$

- $T(\hat{\alpha} - \alpha)$ has a nonnormal limiting distribution. This is different from the usual $\sqrt{T}-$asymptotics.

Cointegration

- There are various methods for estimating α efficiently.
- How can we know x and y are cointegrated?
  Step 1: Run OLS and get residuals $\{\hat{u}_t\}$.
  Step 2: Test for a unit root using $\{\hat{u}_t\}$. (If unit root, non-cointegration. If no unit root, cointegration.)
- More sophisticated methods use the vector autoregression in order to test for the presence of cointegration.