Introduction to Econometrics

Chapter 18: Testing for a Unit Root

In Choi

Sogang University

Autoregressive integrated moving average (ARIMA) model

- Popularized by Box and Jenkins (1976).
- If d is nonnegative integer, $\{X_t\}$ is an $ARIMA(p, d, q)$ process if

$$r_t = (1 - B)^d X_t$$

is an $ARMA(p, q)$ process.

Autoregressive integrated moving average (ARIMA) model

- Many economic time series are well represented by the $ARIMA(p, 1, q)$ model (See Nelson and Plosser, 1982, Journal of Monetary Economics). Examples are GNP,CPI,interest rate,exchange rate, etc.
- $\{r_t\}$ is said to have a stochastic trend. This is because $\{r_t\}$ does not show quickly fluctuating behavior.

Autoregressive integrated moving average (ARIMA) model

- How do we know that $d = 1$? Perform unit root test.
- See Choi (2015) "Almost all about unit roots: foundations, developments and applications," Cambridge University Press for the literature on unit roots.
- Consider the AR(1) model

$$r_t = \phi r_{t-1} + a_t, \;\; a_t \sim iid(0, \sigma^2).$$

- Let $\widehat{\phi}$ be the OLS estimator of ϕ. When $|\phi| < 1$,

$$\sqrt{T}(\widehat{\phi} - \phi) \xrightarrow{d} N(0, 1 - \phi^2)$$

as $T \to \infty$.

- Thus,

$$t(\phi) = \frac{\widehat{\phi} - \phi}{\sqrt{\hat{\sigma}^2 \left(\sum r_{t-1}^2\right)^{-1}}} \xrightarrow{d} N(0, 1),$$

where $\hat{\sigma}^2 = \frac{1}{T-1} \sum_{t=2}^{T} (r_t - \widehat{\phi} r_{t-1})^2$.

- However, when $\phi = 1$,

$$T(\widehat{\phi} - 1) \xrightarrow{d} \left(\int_0^1 W^2(r)\, \mathrm{d}r \right)^{-1} \int_0^1 W(r)\, \mathrm{d}W(r),$$

  where $W(r)$ is Brownian motion.

- A continuous-time stochastic process, $\{W(r), 0 \leq r \leq 1\}$, is called Brownian motion or a Wiener process if it satisfies the following conditions.

(i) $W(0) = 0$, almost surely.
(ii) For $0 \leq t_0 \leq t_1 \leq \ldots \leq t_k$, $W(t_1) - W(t_0), \ldots, W(t_k) - W(t_{k-1})$ are independent.
(iii) $W(t) - W(s)$ $(t > s)$ follows $\mathbf{N}(0, t - s)$.

and

- $\int_0^1 W(r)\,\mathrm{d}W(r)$ is a stochastic integral. Note that

$$\int_0^1 W(r)\,\mathrm{d}W(r) = \frac{1}{2}(W^2(1) - 1) = \frac{1}{2}(\chi^2(1) - 1.$$

- In addition,

$$t(1) = \frac{\widehat{\phi} - 1}{\sqrt{\hat{\sigma}^2(\sum r_{t-1}^2)^{-1}}} \xrightarrow{d} \frac{\int_0^1 W(r)\,\mathrm{d}W(r)}{\sqrt{\int_0^1 W^2(r)\,\mathrm{d}r}}.$$

Autoregressive integrated moving average (ARIMA) model

- The distribution of $T\left(\hat{\phi}-1\right)$ and $t(1)$ are tabulated in Wayne Fuller's "Introduction to Statistical Time Series" (1976,Wiley).
- These are unit root test statistics. Critical values of these test statistics are taken from the LHS tails of the distributions when $H_1 : |\phi| < 1$.

Autoregressive integrated moving average (ARIMA) model

- Alternatively, we may write the model as

$$\Delta r_t = \lambda r_{t-1} + a_t, \;\; a_t \sim iid(0, \sigma^2)$$

and test the null hypothesis $H_0 : \lambda = 0$. The test statistics are

$$T\hat{\lambda} \text{ and } \frac{\widehat{\lambda}}{\sqrt{\hat{\sigma}^2 (\sum r_{t-1}^2)^{-1}}}$$

Autoregressive integrated moving average (ARIMA) model

- When

$$r_t - \mu = \phi(r_t - \mu) + u_t,$$

or

$$r_t = \mu(1-\phi) + \phi r_{t-1} + u_t,$$

$\widehat{\phi}$ also has a nonnormal distribution in the limit if $\phi = 1$.

- The unit root test statistics for this model are:

$$T(\widehat{\phi} - 1) \text{ and } \frac{\widehat{\phi} - 1}{\sqrt{\hat{\sigma}^2 \left(\sum_{t=2}^{T} (r_{t-1} - \bar{r}_{-})^2 \right)^{-1}}},$$

where $\widehat{\phi}$ is the OLS estimator.

- Their limiting distributions are, respectively,

$$\frac{\int_0^1 \overline{W}(r)\,\mathrm{d}W(r)}{\int_0^1 \overline{W}^2(r)\,\mathrm{d}r} \text{ and } \frac{\int_0^1 \overline{W}(r)\,\mathrm{d}W(r)}{\sqrt{\int_0^1 \overline{W}^2(r)\,\mathrm{d}r}},$$

where $\overline{W}(r) = W(r) - \int_0^1 W(r)\,\mathrm{d}r$.

Autoregressive integrated moving average (ARIMA) model

- An AR(p) model

$$r_t = \phi_1 r_{t-1} + ... + \phi_p r_{t-p} + a_t, \;\; a_t \sim WN(0, \sigma^2)$$

can be written as

$$\Delta r_t = \lambda r_{t-1} + \sum_{j=2}^{p} w_j \Delta r_{t-j+1} + a_t, \;\; a_t \sim WN(0, \sigma^2)$$

where the values of $\lambda = \phi_1 + ... + \phi_p - 1$ and $w_j = -\sum_{k=j}^{p} \phi_k$.

Autoregressive integrated moving average (ARIMA) model

- When there is a unit root[1], $\phi_1 + ... + \phi_p = 1$. Thus, the null of a unit root can be tested by testing $\lambda = 0$.
- The t-test for this null hypothesis is called the augmented Dickey-Fuller test. Its asymptotic distribution is $\frac{\int_0^1 W(r)\,\mathrm{d}W(r)}{\sqrt{\int_0^1 W^2(r)\,\mathrm{d}r}}$.

[1] That is, one of the roots of the polynomial equation $1 - \phi_1 z - ... - \phi_p z_p = 0$ is 1.