Econometrics

Chapter 10: Basic Concepts and Models in Time Series Analysis

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Stochastic Processes

Definition

A stochastic process is a family of random variables $\{X_t(\omega),\ t = 1, 2, ...\}$.

- For each fixed t, $X_t(\omega)$ is a random variable.
- For fixed ω, $X_t(\omega)$ is a real valued function of t (realizations or sample–paths of the process $\{X_t, t = 1, 2, ...\}$)
- In time series analyses, we model data as realizations of stochastic processes.
- In basic statistics, we model data as realizations of a random variable.

Example

$X_i \sim iid\ N(\mu,\ \sigma^2)$.

Stochastic Processes

Example

Random walk

$$S_t = \sum_{i=1}^{t} X_i, \;\; t \geq 1$$

$X_i's$ are iid random variables.

Example

Branching processes

$$X_{t+1} = \sum_{j=1}^{X_t} Z_{t,j}$$

$Z_{t,j}$ represents the number of offspring of the j^{th} individual born in generation t.

Weak stationarity and strict stationarity

Definition

If $\{X_t, t = 1, 2...\}$ is a stochastic process such that $Var(X_t) < \infty$ for each t, the autocovariance function $\gamma_X(\cdot, \cdot)$ of $\{X_t\}$ is defined by

$$\gamma_X(r, s) = Cov(X_r, X_s) = E[(X_r - EX_r)(X_s - EX_s)], \ \ r, s = 1, 2, ...$$

- Autocovaraince function shows how X_t is related to its future and past.
- Most economic time series are correlated over time.

Weak stationarity and strict stationarity

Example

Let $X_t = Z_t + Z_{t-1},\ Z_t \sim iid\ \left(0, \sigma^2\right)$.

$$
\begin{aligned}
\gamma(r,s) &= E\left[X_r - EX_r\right]\left[X_s - EX_s\right] \\
&= E\left(Z_r + Z_{r-1}\right)\left(Z_s + Z_{s-1}\right) \\
&= \begin{cases} 2\sigma^2, & r = s \\ \sigma^2, & |r-s| = 1 \\ 0, & o.w. \end{cases}.
\end{aligned}
$$

Weak stationarity and strict stationarity

Definition

The time series $\{X_t, t = 1, 2, ...\}$ is said to be weakly stationary if

1. $E\left|X_t^2\right| < \infty$ for all t.
2. $EX_t = m$ for all t.
3. $\gamma_X(r, s) = \gamma_X(r + t, s + t)$ for all r, s, t.

- Weak stationarity = covariance stationarity, stationarity in the wide sense, second–order stationarity.
- Weak stationarity is the most basic assumption in time series analysis. Without it, it is hard to proceed further.

Weak stationarity and strict stationarity

- For weakly stationary processes,

$$\gamma_X(r,s) = \gamma_X(r-r, s-r) = \gamma_X(0, s-r) = \gamma_X(h), \; h = s-r.$$

  That is, the covariance function depends only on the time time difference.

- Autocorrelation function

$$\rho_X(h) = \gamma_X(h)/\gamma_X(0) \text{ for all } t, h = 1, 2, \ldots.$$

Weak stationarity and strict stationarity

Example

$X_t = Z_t + Z_{t-1},\ Z_t \sim iid\left(0, \sigma^2\right)$

$$Cov\left(X_{t+h}, X_t\right) = \begin{cases} 2\sigma^2, & h = 0 \\ \sigma^2, & h = \pm 1 \\ 0, & |h| > 1 \end{cases} .$$

$$\text{(does not depend on } t\text{)}$$

$$\sim \quad \text{weakly stationary}$$

$$\rho_X(h) = \begin{cases} 1, & h = 0 \\ 1/2, & h = \pm 1 \\ 0, & |h| > 1 \end{cases} .$$

Weak stationarity and strict stationarity

Example

$S_t = \sum_{i=1}^{t} X_i, \; X_i \sim iid\left(0, \sigma^2\right)$

$$\begin{aligned} Cov\left(S_{t+h}, S_t\right) &= E\left(\sum_{i=1}^{t+h} X_i\right)\left(\sum_{i=1}^{t} X_i\right) \\ &= \sigma^2 t. \\ &\sim \text{nonstationary} \end{aligned}$$

Definition

The time series $\{X_t, t = 1, 2, ...\}$ is said to be strictly stationary if the joint distribution of $(X_{t_1}, \cdots, X_{t_k})'$ and $(X_{t_1+h}, \cdots, X_{t_k+h})'$ are the same for all positive integers k and for all $t_1, \cdots, t_k, \; h$.

- strict stationarity with finite second moments $\Rightarrow$ stationarity.
- Gaussian (multivariate normal) weak stationarity $\Rightarrow$ strict stationarity.

Properties of autocovariance function

- Let $\gamma(h)$ be an autocovariance function of a weakly stationary process X_t.

1. $\gamma(0) \geq 0$
2. $|\gamma(h)| \leq \gamma(0)$ for all $h = 0, \pm 1, \pm 2, \ldots$

$$
\begin{aligned}
|\gamma(h)| = |Cov(X_{t+h}, X_t)| &\leq (VarX_{t+h})^{1/2} (VarX_t)^{1/2} \\
&= \gamma(0)
\end{aligned}
$$

3. $\gamma(-h) = \gamma(h)$ for all $h = 0, \pm 1, \pm 2, \ldots$

$$
\gamma(-h) = EX_t X_{t-h} = EX_{t-h} X_t = EX_t X_{t+h} = \gamma(h).
$$

Properties of autocovariance function

Definition

The sample autocovariance function of the observations $\{x_1, \cdots, x_n\}$ for a weakly stationary time series $\{X_t\}$ is defined by

$$\hat{\gamma}(h) = n^{-1} \sum_{t=1}^{n-h} (X_{t+h} - \bar{X})(X_t - \bar{X}), \ 0 \leq h \leq n,$$

and $\hat{\gamma}(h) = \hat{\gamma}(-h)$, where $\bar{X} = n^{-1} \sum_{t=1}^{n} X_t$.

- $\hat{\rho}(h) \equiv \hat{\gamma}(h) / \hat{\gamma}(0)$.
- $\hat{\rho}(1) > 0$ reflects a tendency for successive observations to lie on the same side of the mean, while $\hat{\rho}(1) < 0$ reflects a tendency for successive observations to lie on the opposite sides of the mean.

ARMA Processes

Definition

The process $\{Z_t\}$ is said to be white noise with mean 0 and variance σ^2 $\left(\{Z_t\} \sim WN\left(0, \sigma^2\right)\right)$ iff $EZ_t = 0$ for all t and

$$EZ_t Z_s = \begin{cases} 0, & t \neq s \\ \sigma^2, & t = s \end{cases}.$$

Example

An iid process with zero mean is a white noise process.

ARMA Processes

Definition

The process $\{X_t, t = 0, \pm 1, \pm 2, \cdots\}$ is said to be an $ARMA\,(p, q)$ process (autoregressive moving average process of order p and q) if

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

where $Z_t \sim WN\left(0, \sigma^2\right)$.

- In applications, $X_t, .., X_{t-p}$ are observed time series. But $Z_t, ..., Z_{t-q}$ are not observed.

ARMA Processes

- If $X_t \sim ARMA(p_1, q_1)$ and $Y_t \sim ARMA(p_2, q_2)$, $X_t + Y_t \sim ARMA(P, Q)$ where $P = p_1 + p_2$ and $Q = \max(p_1 + q_2, p_2 + q_1)$.
- For example, if $X_t \sim AR(1)$ and $Y_t \sim AR(1)$, $X_t + Y_t \sim ARMA(2, 1)$.
- Aggregation of AR processes results in an ARMA process.
- ARMA model is parsimonious compared to the AR model with many lagged variables. But it requires nonlinear optimization for estimation unlike the AR model.

ARMA Processes

- We may write the *ARMA* process as

$$\begin{aligned} \phi(B) X_t &= \theta(B) Z_t \\ \phi(B) &= 1 - \phi_1 B - \cdots - \phi_p B^p \\ \theta(B) &= 1 + \theta_1 B + \cdots + \theta_q B^q \\ B^r y_t &= y_{t-r}. \end{aligned}$$

Example

$MA(1)$ process

$$X_t = Z_t + \theta Z_{t-1}, \; Z_t \sim WN\left(0, \sigma^2\right).$$

Example

$AR(1)$ process

$$X_t = \phi X_{t-1} + Z_t.$$

ARMA Processes

Definition

An ARMA(p, q) process defined by the equations $\phi(B) X_t = \theta(B) Z_t$ is said to be invertible if there exists a sequence of constants $\{\pi_j\}$ such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}, \ \ t = 0, \pm 1, \cdots .$$

- In other words, an ARMA process is invertible if it can be written as an AR(∞) process.

Theorem

(Invertibility) Let $\{X_t\}$ *be an ARMA*(p, q) *process for which the polynomials* $\phi(\cdot)$ *and* $\theta(\cdot)$ *have no common zeros. Then* $\{X_t\}$ *is invertible iff* $\theta(z) \neq 0$ *for all* $z \in \mathbb{C}$ *such that* $|z| \leq 1$*. The coefficient* π_j *are determined by the relation*

$$\pi(z) = \sum_{j=0}^{\infty} \pi_j z^j = \phi(z) / \theta(z), \ |z| \leq 1.$$

ARMA Processes

Example

$$\begin{cases} X_t = Z_t - \theta Z_{t-1}, Z_t \sim WN\left(0, \sigma^2\right) \\ |\theta| < 1. \end{cases}$$

$$\begin{aligned} X_t &= Z_t - \theta Z_{t-1} \\ (+\theta) X_t &= (+\theta) Z_{t-1} - \theta^2 Z_{t-2} \\ \theta^2 X_{t-2} &= \theta^2 Z_{t-2} - \theta^3 Z_{t-3} \\ &\vdots \\ \sum_{i=0}^{\infty} X_{t-i}\theta^i &= Z_t. \end{aligned}$$

Because $\sum_{i=0}^{\infty} \left|\theta^i\right| < \infty$, Z_t is invertible.

ARMA Processes

Theorem

(Weak stationarity) If $\phi(z) \neq 0$ *for all* $z \in \mathbb{C}$ *such that* $|z| \leq 1$*, the ARMA process* X_t *is weakly stationary.*

ARMA Processes

Example

If $X_t = \phi X_{t-1} + Z_t$ and $|\phi| < 1$,

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j},$$

and X_t is stationary.

Example

If $X_t = \phi X_{t-1} + Z_t,\ \phi = 1,\ \{X_t\}$ is not stationary.

Detrending and deseasonalizing time series

- Time series X_t is usually represented as

$$X_t = m_t + S_t + Y_t,$$

where m_t is a trend component, S_t a seasonal component and Y_t a random component.

Example

GNP
m_t : growth component, S_t : seasonal component, Y_t : economic fluctuation

1. Linear regression method
   Let $m_t = a_0 + a_1 t + \cdots + a_p t^p$. Run OLS regression

$$X_t = \hat{m}_t + \hat{Y}_t$$

   to estimate $a_0, a_1, \cdots, a_p$. The detrended series is

$$X_t - \hat{m}_t = \hat{Y}_t.$$

   In practice, we usually use $p = 1$.

2. Differencing

$$\Delta X_t = X_t - X_{t-1}$$

Example

$$\begin{aligned} X_t &= at + Y_t \\ \Rightarrow \Delta X_t &= a + \Delta Y_t \end{aligned}$$

- The Bureau of Census X–12 ARIMA methods.

- Define

$$\Delta_d X_t = X_t - X_{t-d}.$$

Applying Δ_d, we may detrend and deseasonalize a data

$$\begin{aligned} d &= 4;\ \text{quarterly data} \\ d &= 12;\ \text{monthly data.} \end{aligned}$$

Example

Suppose that

$$X_t = \mu t + S_t + Y_t,$$

where S_t is the seasonal component with the property $S_t = S_{t-d}$ and Y_t is a weakly stationary process. Then

$$\begin{aligned} \Delta_d X_t &= (\mu t - \mu(t-d) + S_t - S_{t-d} + Y_t - Y_{t-d}) \\ &= \mu d + Y_t - Y_{t-d}. \end{aligned}$$

This is a weakly stationary process.