Advanced Econometrics

Chapter 10: Specification Tests

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White's test for heteroskedasticity

Main idea

- For a model with heteroskedasticity,

$$y_i = x_i' \beta + \varepsilon_i,$$

we have

$$E(b) = \beta \text{ and } Var(b \mid X) = (X'X)^{-1} X' \Sigma X (X'X)^{-1}$$

where

$$\Sigma = diag\left[\sigma_1^2, \cdots, \sigma_n^2\right].$$

- We may express

$$X'\Sigma X = \sum_{i=1}^{n} \sigma_i^2 x_i x_i'.$$

- Suppose that

$$n^{-1}X'X = n^{-1}\sum_{i=1}^{n} x_i x_i' \to Q,$$

a finite and nonsingular matrix.

White's test for heteroskedasticity

Main idea

- Then

$$n^{-1}X'\Sigma X = n^{-1}\sum_{i=1}^{n}\sigma_i^2 x_i x_i'$$

can be estimated consistently by

$$\hat{V}_n = n^{-1}\sum_{i=1}^{n} e_i^2 x_i x_i'$$

where

$$e_i = y_i - x_i' b.$$

(Note that $b \xrightarrow{p} \beta$ even in the presence of heteroskedasticity.)

White's test for heteroskedasticity

Main idea

- If there is no heteroskedasticity $\left(\sigma_1^2 = \cdots = \sigma_n^2\right)$, $n^{-1}X'\Sigma X$ is consistently estimated by either $\hat{\sigma}^2\left(n^{-1}X'X\right)$ where $\hat{\sigma}^2 = n^{-1}\left(y - X'b\right)\left(y - X'b\right)$ or $\hat{V}_n$. Thus, comparing $\hat{V}_n$ and $\hat{\sigma}^2\left(n^{-1}X'X\right)$ provides an indicator of heteroskedasticity. When there is no heteroskedasticity,

$$\hat{V}_n - \hat{\sigma}^2\left(n^{-1}X'X\right) \xrightarrow{p} 0.$$

Otherwise,

$$\hat{V}_n - \hat{\sigma}^2\left(n^{-1}X'X\right) \overset{p}{\nrightarrow} 0.$$

- White suggests to use

$$WH = nD\left(b, \hat{\sigma}^2\right)' \hat{B}^{-1} D\left(b, \hat{\sigma}^2\right),$$

where

$$D\left(b, \hat{\sigma}^2\right) = n^{-1} \sum_{i=1}^{n} \Psi_i \left(e_i^2 - \hat{\sigma}^2\right);$$

$$\hat{B} = n^{-1} \sum_{i=1}^{n} \left(e_i^2 - \hat{\sigma}^2\right)^2 \left(\Psi_i - \hat{\Psi}\right)' \left(\Psi_i - \hat{\Psi}\right);$$

White's test for heteroskedasticity

The test statistic

- Ψ_i is the $K(K+1)/2 \times 1$ vector containing the element of the lower triangle of the matrix $x_i x_i'$, and

$$\hat{\Psi} = n^{-1} \sum_{i=1}^{n} \Psi_i.$$

If $k = 2$, $\Psi_i = \left(x_{1i}^2, x_{1i}x_{2i}, x_{2i}^2\right)'$.
Under the null of no heteroskedasticity,

$$WH \xrightarrow{d} \chi^2_{K(K+1)/2}$$
$$(K : \text{no of regressors}).$$

- Note that $D\left(b, \hat{\sigma}^2\right)$ is the vectorized form of $\hat{V}_n - \hat{\sigma}^2\left(n^{-1}X'X\right)$.
- The limiting distribution of WH depends on the number of regressors in the model.

Lagrange multiplier test for heteroskedasticity

Reference

- Breusch and Pagan (1979) "A Simple Test for Heteroskedasticity and Random Coefficient Variation." Econometrica.

- $$y_i = x_i'\beta + \varepsilon_i,\ \{x_i\} \text{ is a sequence of constant vectors.}$$
  $$\varepsilon_i \text{ are independent and have a normal distribution} N(0, \sigma_i^2)$$
  $$\begin{aligned}\sigma_i^2 &= h\left(\alpha_0 + z_i'\alpha\right),\ \alpha = (\alpha_1, \cdots, \alpha_p)', \\ &\quad \{z_i\} \text{ is a sequence of constant vectors}\end{aligned}$$
  $$H_0 : \alpha_1 = \cdots = \alpha_p = 0\ \ (\text{no heteroskedasticity})$$

- The Lagrangian is given as

$$\begin{aligned} & \ln L + \lambda'\alpha \\ = \;& const. - \frac{1}{2}\sum_{i=1}^{n}\ln(\sigma_i^2) - \frac{1}{2}\sum_{i=1}^{n}(y_i - x_i'\beta)^2/\sigma_i^2 + \lambda'\alpha. \end{aligned}$$

- The first-order condition evaluated under the null restriction is written as

$$-\frac{1}{2}\sum_{i=1}^{n}\frac{1}{\sigma^{*2}}h'(\alpha_0^*)z_i + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\sigma^{*4}}(y_i - x_i'\beta^*)^2 h'(\alpha_0^*)z_i + \lambda^* = 0.$$

- This gives

$$\lambda^* = \frac{1}{2}\frac{h'(\alpha_0^*)}{h(\alpha_0^*)}\sum_{i=1}^{n}\left(1-\frac{e_i^{*2}}{\sigma^{*2}}\right)z_i \ \left(h(\alpha_0^*)=\sigma^{*2}\right)$$
$$= k^*\sum_{i=1}^{n} f_i^* z_i, \ say.$$

Here e_i^* is the OLS residual.

- Under the null hypothesis, $f_i^* = f_i + o_p(n^{-1/2})$ where $f_i = 1 - \frac{\varepsilon_i^2}{\sigma^2}$. Thus,

$$\frac{\lambda^*}{k^*\sqrt{n}} - \frac{\sum_{i=1}^n f_i z_i}{\sqrt{n}} \xrightarrow{p} 0.$$

But

$$\frac{\sum_{i=1}^n f_i z_i}{\sqrt{n}} \xrightarrow{d} N\left(0, 2\lim \frac{\sum_{i=1}^n z_i z_i'}{n}\right) \quad (var(f_i) = 2)$$

and

$$k^* \xrightarrow{p} k = \frac{h'(\alpha_0)}{h(\alpha_0)}.$$

- We deduce from these

$$\frac{\lambda^*}{\sqrt{n}} \xrightarrow{d} N\left(0, 2k^2 \lim \frac{\sum_{i=1}^n z_i z_i'}{n}\right).$$

- The LM test is defined by

$$\begin{aligned} LM &= \frac{\lambda^{*\prime}}{\sqrt{n}}\left(2k^{*2}\frac{\sum_{i=1}^n z_i z_i'}{n}\right)^{-1}\frac{\lambda^*}{\sqrt{n}} \\ &= \frac{1}{2}\left(\sum_{i=1}^n z_i f_i^*\right)'\left(\sum_{i=1}^n z_i z_i'\right)^{-1}\left(\sum_{i=1}^n z_i f_i^*\right). \end{aligned}$$

  Remarkably, this does not depend on k^* or equivalently on $h(\cdot)$.

- As $n \longrightarrow \infty$,

$$LM \xrightarrow{d} \chi_p^2.$$

- Alternatively, we may use

$$LM' = \left(\sum_{i=1}^{n} z_i f_i^*\right)' \left(\sum_{i=1}^{n} z_i z_i'\right)^{-1} \left(\sum_{i=1}^{n} z_i f_i^*\right) / \left(\frac{1}{n}\sum_{i=1}^{n} f_i^{*2}\right).$$

Note that $\frac{1}{n}\sum_{i=1}^{n} f_i^{*2}$ estimates $var(f_i)$, which was set at 2 for LM. This version of LM test is expected to be more robust to non-normal errors.

- The LM test is independent of the functional form $h(\cdot)$.
- We need to specify exogenous variable z_i to apply the LM test. We usually use $z_i = x_i$.

Lagrange multiplier test for a nonlinear model

- Consider the nonlinear regression model

$$y_t = g\left(X_t; \theta\right) + \varepsilon_t \equiv g_t + \varepsilon_t$$

  where $\varepsilon_t \sim iid\ N\left(0, \sigma^2\right)$ and g_t is independent of ε_t.
- Partition $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \begin{matrix} p \\ k-p \end{matrix}$.
- We are interested in testing the null hypothesis $H_0 : \theta_1 = \theta_{10}$.

Lagrange multiplier test for a nonlinear model

- The log–likelihood function is written as

$$L = const - \frac{T}{2}\log\sigma^2 - \frac{1}{2}\frac{\varepsilon'\varepsilon}{\sigma^2}.$$

- The score and information matrix are,, respectively,

$$D = \frac{\partial L}{\partial \theta} = \sigma^{-2}G'\varepsilon \text{ and } I = E\left(\frac{\partial L}{\partial \theta}\right)\left(\frac{\partial L}{\partial \theta}\right)' = \sigma^{-2}EG'G,$$

where $G = \left[\partial g_t / \partial \theta_k\right]_{(t,k)}$.

Lagrange multiplier test for a nonlinear model

- The LM statistic is based upon

$$\tilde{\sigma}^{-2}\tilde{e}'\tilde{G}'\left[\tilde{E}\left(G'G\right)\right]^{-1}\tilde{G}'\tilde{e}$$

  where $\tilde{e}$ are the residuals from the model after fitting the $K-p$ estimates of θ_2, while restricting $\theta_1 = \theta_{10}$, and $\tilde{E}\left(G'G\right)$ denotes that $EG'G$ is evaluated at the restricted estimates $\tilde{\theta}_1 = \theta_{10}$ and $\tilde{\theta}_2$.

- Consider the regression model

$$y = X\beta + u; \;\; u = u_{-j}\rho_j + \varepsilon$$

  where $\varepsilon \sim N\left(0, \sigma^2 I\right)$ and X is independent of u.
- The transformed model is

$$y = \rho_j y_{-j} + \left(X - X_{-j}\rho_j\right)\beta + \varepsilon \equiv g + \varepsilon.$$

- The null of no autocorrelation is written as

$$H_0 : \rho_j = 0.$$

- We have under this restriction

$$\tilde{e} = \hat{u} \ \text{(OLS residuals)}$$

$$\frac{\partial g}{\partial \rho_j} = (y_{-j} - X_{-j}\beta)' = u'_{-j} \text{ and } \frac{\partial g}{\partial \beta} = \left(X - X_{-j}\rho_j\right)'.$$

- Hence, $\tilde{G} = \begin{pmatrix} \hat{u}_{-j} & X \end{pmatrix}$
- The score vector is $\tilde{D} = \frac{\partial L}{\partial \theta} = \tilde{\sigma}^{-2} \tilde{G}' \tilde{e}$, from which we obtain

$$\begin{aligned} \tilde{D}_1 &= \frac{\partial L}{\partial \rho_j} = \hat{\sigma}^{-2} \hat{u}'_{-j} \hat{u} = \frac{\hat{u}'_{-j} \hat{u}}{\hat{u}' \hat{u} / T} \\ &= T \cdot \frac{\hat{u}'_{-j} \hat{u} / T}{\hat{u}' \hat{u} / T} = T \gamma_j. \; (T \times \text{autocorrelation function}) \end{aligned}$$

- From $\tilde{I} = \hat{\sigma}^{-2}\tilde{G}'\tilde{G}$, we obtain

$$\tilde{I}_{11} = \left[\hat{\sigma}^{-2}\left(\hat{u}_{-j}'\hat{u}_{-j} - \hat{u}_{-j}X\left(X'X\right)^{-1}X'\hat{u}_{-j}\right)\right]^{-1}.$$

- Thus, $LM =$ $T\gamma_j^2\left[\hat{\sigma}^{-2}T^{-1}\hat{u}_{-j}'\hat{u}_{-j} - \hat{\sigma}^{-2}\left(T^{-1}\hat{u}_{-j}X\right)\left(T^{-1}X'X\right)^{-1}\left(T^{-1}X'\hat{u}_{-j}\right)\right]^{-1}$
- Clearly, we have $LM \xrightarrow{d} \chi_1^2$.

Lagrange multiplier test for a nonlinear model

Testing for autocorrelation: autoregressive alternatives

- If X is strictly exogenous and the null is true,

$$\hat{\sigma}^{-2}T^{-1}\hat{u}'_{-j}\hat{u}_{-j} \xrightarrow{p} 1 \text{ and } \hat{\sigma}^{-2}T^{-1}\hat{u}'_{-j}X \xrightarrow{p} 0,$$

  so that the LM statistic becomes $T\gamma_j^2$ asymptotically.
- The LM test can be used even when X involves lagged dependent variable.
- The normality assumption is not important for asymptotic results.

- The LM test can easily be extented to test $H_0 : \rho_1 = \rho_2 = \cdots = \rho_m$ in $u_t = \rho_1 u_{t-1} + \cdots + \rho_m u_{t-m} + e_t$. The asymptotic distribution for this null is χ^2_m.

Exercises

1. Consider the random-coefficient regression model

$$y_i = \mu + \beta_i x_i + u_i, \ (i = 1, ..., n),$$

where $\beta_i \sim iid(\beta, \sigma_\beta^2)$; $y_i \sim iid \ N(\mu + \beta x_i, \sigma_\beta^2 x_i^2 + \sigma_u^2)$ $(\sigma_u^2 = Var(u_i))$; and $\{x_i\}$, $\{\beta_i\}$ and $\{u_i\}$ are independent.

a. Derive the score statistic under the null hypothesis $H_0 : \sigma_\beta^2 = 0$.

b. The score statistic from part (a) involves unknown quantities σ_u^2 and $\{u_i\}$. Explain how these can be estimated consistently under the null hypothesis. The score statistic using these estimators will be called the feasible score statistic.

Exercises

c. Derive the limiting distribution of the feasible score statistic from part (b).
d. Using the result from part (c), formulate the score test statistic for the null hypothesis $H_0 : \sigma^2_\beta = 0$ and derive its limiting distribution.
e. What would happen to the test statistic of part (d) if $\sigma^2_\beta > 0$?