Advanced Econometrics

Chapter 9: Robust OLS and IV

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The model and assumptions

- Model

$$y = X\beta + \varepsilon$$

- Assumptions on the error terms

$$(i)\ E[\varepsilon|X] = 0$$

$$(ii)\ E[\varepsilon\varepsilon'|X] = \begin{cases} \sigma^2\Omega \\ \Sigma \end{cases} \quad (\Omega,\ \Sigma > 0).$$

OLS estimation under heteroskedasticity

- Assume heteroskedasticity. That is,

$$\Sigma = \begin{bmatrix} \sigma_1^2 & & & \underset{\sim}{0} \\ & \ddots & & \\ & & \ddots & \\ \underset{\sim}{0} & & & \sigma_n^2 \end{bmatrix}$$

- OLS estimation
  The OLS estimator can be written as

$$b = \beta + \left(X'X\right)^{-1} X'\varepsilon.$$

OLS estimation under heteroskedasticity

1. Unbiasedness
$$E\left[b\right] = E\left[E\left[b|X\right]\right] = \beta.$$
2. Variance–Coviance Matrix
$$\begin{aligned} Var\left[b|X\right] &= E\left[\left(b-\beta\right)\left(b-\beta\right)'|X\right] \\ &= E\left[\left(X'X\right)^{-1}X'\varepsilon\varepsilon'X\left(X'X\right)^{-1}|X\right] \\ &= \left(X'X\right)^{-1}X'\Sigma X\left(X'X\right)^{-1}. \end{aligned}$$

The unconditional variance is
$$E\left[Var\left[b|X\right]\right].$$

If ε is normally distributed,
$$b|X \sim N\left(\beta, \left(X'X\right)^{-1}X'\Sigma X\left(X'X\right)^{-1}\right).$$

OLS estimation under heteroskedasticity

3. Consistency

   Suppose that

$$\frac{X'X}{n} \xrightarrow{p} Q > 0$$

$$\frac{X'\Sigma X}{n} \xrightarrow{p} P > 0.$$

   Then

$$\begin{aligned} Var\left[b|X\right] &= \frac{1}{n}\left(\frac{X'X}{n}\right)^{-1}\frac{X'\Sigma X}{n}\left(\frac{X'X}{n}\right)^{-1} \\ &\xrightarrow{p} 0 \end{aligned}$$

   and

$$Var\left[b\right] \xrightarrow{p} 0.$$

OLS estimation under heteroskedasticity

Using this and Chebyshev's inequality, we have for and $\alpha \in \mathbb{R}^k - \{0\}$ and $\varepsilon > 0$

$$\begin{aligned} P\left[\left|\alpha'(b-\beta)\right| > \varepsilon\right] &\leq \frac{\alpha' E(b-\beta)(b-\beta)'\alpha}{\varepsilon^2} \\ &= \frac{\alpha' Var(b)\alpha}{\varepsilon^2} \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

which implies

$$b \xrightarrow{p} \beta.$$

OLS estimation under heteroskedasticity

4. Asymptotic distribution of b

   Assume (x_i, ε_i) is a sequence of independent observations with

$$\begin{aligned} E\left(\varepsilon\varepsilon'\right) &= diag\left(\sigma_1^2, \cdots, \sigma_n^2\right) \\ &= \Sigma. \end{aligned}$$

   In addition, assume for any $\lambda \in \mathbb{R}^k - \{0\}$ and $\delta > 0$

$$E\left|\lambda' x_i \varepsilon_i\right|^{2+\delta} \leq B \text{ for all } i.$$

   Then, we can apply the CLT for a sequence of independent random variables, with gives

$$\frac{\sum_{i=1}^n \lambda' x_i \varepsilon_i}{\sqrt{n}} \xrightarrow{d} N\left(0, \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{\infty} E\left(\lambda' x_i \varepsilon_i^2 x_i' \lambda\right)\right).$$

OLS estimation under heteroskedasticity

But

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} E\left(\lambda' x_i \varepsilon_i^2 x_i' \lambda\right) &= \frac{1}{n}\sum_{i=1}^{n} EE\left(\lambda' x_i \varepsilon_i^2 x_i' \lambda | X\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} \sigma_i^2 \lambda' E\left(x_i x_i'\right) \lambda \\
&= \frac{1}{n}\lambda' \sum_{i=1}^{n} \sigma_i^2 E\left(x_i x_i'\right) \lambda \\
&\to \operatorname{plim} \frac{1}{n}\lambda' \sum_{i=1}^{n} \sigma_i^2 x_i x_i' \lambda \\
&= \operatorname{plim} \frac{1}{n}\lambda' \left(X'\Sigma X\right) \lambda.
\end{aligned}
$$

OLS estimation under heteroskedasticity

Thus

$$\frac{\sum_{i=1}^{n} x_i \varepsilon_i}{\sqrt{n}} \xrightarrow{d} N\left(0, P\right),$$

where

$$P = \text{plim}\frac{1}{n} X' \Sigma X,$$

and we obtain

$$\sqrt{n}\left(b - \beta\right) \xrightarrow{d} N\left(0, Q^{-1} P Q^{-1}\right).$$

IV estimation under heteroskedasticity

- IV estimation

  The IV estimator is written as

$$b_{IV} = \beta + (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'\epsilon.$$

Assume $(z_i', \varepsilon_i)'$ is a sequence of independent random vectors with

$$E(\varepsilon\varepsilon'|Z) = diag\left(\sigma_1^2, \cdots, \sigma_n^2\right) = \Sigma$$

and $E\left|\lambda' z_i \varepsilon_i\right|^{2+\delta} \leq B$ for all i for any $\lambda \in \mathbb{R}^k - \{0\}$ and $\delta > 0$.

IV estimation under heteroskedasticity

- In addition, assume

$$\frac{Z'Z}{n} \xrightarrow{p} Q_{ZZ} \; (>0)$$
$$\frac{Z'X}{n} \xrightarrow{p} Q_{ZX} \; (\neq 0)$$
$$\frac{X'X}{n} \xrightarrow{p} Q_{XX} \; (>0)$$
$$\frac{Z'\Sigma Z}{n} \xrightarrow{p} Q_{Z\Sigma Z}.$$

- The CLT gives

$$\frac{\sum_{i=1}^{n} z_i \varepsilon_i}{\sqrt{n}} \xrightarrow{d} N\left(0, \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{\infty} E\left(z_i \varepsilon_i^2 z_i'\right)\right) = N(0, Q_{Z\Sigma Z})$$

as before.

IV estimation under heteroskedasticity

- Let

$$Q_{XX.Z} = \left(Q_{XZ} Q_{ZZ}^{-1} Q_{ZX}\right)^{-1} Q_{XZ} Q_{ZZ}^{-1},$$

which is the probability limit of $\frac{1}{n}(X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}$. Then, we obtain

$$\sqrt{n}\,(b_{IV} - \beta) \xrightarrow{d} N\left(0, Q_{XX.Z} Q_{Z\Sigma Z} Q'_{XX.Z}\right).$$

Robust estimation of asymptotic covariance matrices

- We can still use OLS for inference if its variance–covariance matrix

$$\left(X'X\right)^{-1} X'\Sigma X \left(X'X\right)^{-1}$$

  can be estimated.

- Notice that

$$\Sigma = diag\left(\sigma_1^2, \cdots, \sigma_n^2\right).$$

  Obviously, $\sigma_1^2, \cdots, \sigma_n^2$ cannot be estimated. But what we need is to estimate $X'\Sigma X$ not Σ.

- We may write

$$\frac{1}{n}X'\Sigma X = \frac{1}{n}\sum_{i=1}^{n}\sigma_i^2 x_i x_i'.$$

Robust estimation of asymptotic covariance matrices

- This and

$$\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2 x_i x_i'$$

have the same probability limit by the LLN. We replace ε_i^2 with e_i^2 (OLS residuals) and, then, have

$$\begin{aligned} & \frac{1}{n}\sum_{i=1}^{n} e_i^2 x_i x_i' \\ = & \frac{1}{n}\sum_{i=1}^{n}\left[\varepsilon_i - x_i'\left(\hat{\beta}-\beta\right)\right]^2 x_i x_i' \\ = & \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2 x_i x_i' + o_p(1). \end{aligned}$$

(See White, 1980, Econometrica for details)

Robust estimation of asymptotic covariance matrices

- Thus $\frac{1}{n}\sum_{i=1}^{n} e_i^2 x_i x_i'$ consistently estimate $\frac{1}{n}X'\Sigma X$, and the estimated asymptotic variance–covariance matrix b is

$$\left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} e_i^2 x_i x_i'\right)\left(\frac{1}{n}X'X\right)^{-1} \xrightarrow{p} Q^{-1}PQ^{-1}.$$

- We can use this result for hypothesis testing. Suppose that the null hypothesis is

$$H_0 : R\beta = r.$$

Robust estimation of asymptotic covariance matrices

- Then, Wald test is defined by

$$W = (Rb - r)' \left[R \left(X'X \right)^{-1} \sum_{i=1}^{n} e_i^2 x_i x_i' \left(X'X \right)^{-1} R' \right]^{-1} (Rb - r),$$

(heteroskedasticity robust Wald test)

and as $n \longrightarrow \infty$

$$W \xrightarrow{d} \chi^2 (J), \; J = rank (R).$$

- This follows because

$$\begin{aligned} W &= \sqrt{n} (Rb - r)' \left[R \left(\frac{X'X}{n} \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^{n} e_i^2 x_i x_i' \right) \left(\frac{X'X}{n} \right)^{-1} R' \right]^{-1} \\ &\quad \times \sqrt{n} (Rb - r) \\ &\quad \xrightarrow{d} N (0, I_J)' N (0, I_J) = \chi^2 (J). \end{aligned}$$

Robust estimation of asymptotic covariance matrices

- If the null hypothesis is $H_0 : \beta_k = \beta_k^0$, use the t-ratio

$$t = \frac{b_k - \beta_k^0}{\sqrt{V_{kk}}}$$

where

$$V = \left(X'X\right)^{-1} \sum_{i=1}^{n} e_i^2 x_i x_i' \left(X'X\right)^{-1}.$$

As $n \to \infty$

$$t \xrightarrow{d} N\left(0, 1\right).$$

This is celebrated White's heteroskedasticity robust t-ratio.