Advanced Econometrics

Chapter 7: Structural change

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Structural change

- Examples of events that led to structural change
  - Oil shock of 1973
  - Asian currency crisis
- Some relevant question
  - Did the Finnish metal product and machinery industry become more energy-efficient after the oil shock? (cf. Ilmakunnas and Törma̋, JAE, 1994)
  - Did the financial industry in Asia become more efficient after the currency crisis?
- Letting

$$\begin{aligned} Y_1, X_1 &: \text{data for period } 1 \\ Y_2, X_2 &: \text{data for period } 2, \end{aligned}$$

we want to test whether coefficient vector undergoes changes from period 1 to period 2.

Structural change

- Model

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

- OLS estimator

$$\begin{aligned} b = (X'X)^{-1} X'y &= \begin{pmatrix} X_1'X_1 & 0 \\ 0 & X_2'X_2 \end{pmatrix}^{-1} \begin{pmatrix} X_1'y_1 \\ X_2'y_2 \end{pmatrix} \\ &= \begin{pmatrix} (X_1'X_1)^{-1} & 0 \\ 0 & (X_2'X_2)^{-1} \end{pmatrix} \begin{pmatrix} X_1'y_1 \\ X_2'y_2 \end{pmatrix} \\ &= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \end{aligned}$$

Structural change

- The null of no structural change can be written as

$$H_0 : R\beta = r$$

where

$$R = \begin{bmatrix} I & -I \end{bmatrix} \text{ and } r = 0.$$

- We may use Wald test for this null hypothesis. The Wald test statistic is defined as

$$W = (Rb - r)' \left(Rs^2 \left(X'X \right)^{-1} R' \right)^{-1} (Rb - r) \xrightarrow{d} \chi^2 (K)$$

- Let

$$\begin{aligned} y_1 &= Z_1 \underset{K_1 \times 1}{\gamma} + X_1 \underset{K_2 \times 1}{\beta_1} + \varepsilon_1 \\ y_2 &= Z_2 \gamma + X_2 \beta_2 + \varepsilon_2 \end{aligned}$$

- We want to test $H_0 : \beta_1 = \beta_2$. The model is written as

$$\begin{aligned} y &= \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \gamma + \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \beta + \varepsilon \\ &= \begin{pmatrix} Z_1 & X_1 & 0 \\ Z_2 & 0 & X_2 \end{pmatrix} \begin{pmatrix} \gamma \\ \beta \end{pmatrix} + \varepsilon \\ &= X\delta + \varepsilon. \end{aligned}$$

- The null hypothesis is written as $H_0 : R\delta = r$ where

$$R = \begin{bmatrix} \underset{K_1}{0} & \underset{K_2}{I} & \underset{K_3}{-I} \end{bmatrix} \text{ and } \underset{K_2 \times 1}{r} = 0.$$

- Thus, we proceed as before. In this case,

$$W \xrightarrow{d} \chi^2 \left(K_2 \right).$$

- Suppose that $\varepsilon_{1i} \sim iid\left(0, \sigma_1^2\right)$ and $\varepsilon_{2i} \sim iid\left(0, \sigma_2^2\right)$ $\left(\sigma_1^2 \neq \sigma_2^2\right)$. In additions, assume $\left(X_1, \varepsilon_1\right)$ and $\left(X_2, \varepsilon_2\right)$ are independent. Wald test for the null hypothesis

$$H_0 : \beta_1 = \beta_2$$

is defined by

$$W = \left(\hat{\beta}_1 - \hat{\beta}_2\right)' \left[s_1^2 \left(X_1' X_1\right)^{-1} + s_2^2 \left(X_2' X_2\right)^{-1}\right]^{-1} \left(\hat{\beta}_1 - \hat{\beta}_2\right).$$

As $n \to \infty$, $W \xrightarrow{d} \chi^2\left(K_2\right)$.

The formula follows because under the null

$$E\left(\hat{\beta}_1 - \hat{\beta}_2\right) = 0$$

$$Asy.Var\left(\hat{\beta}_1 - \hat{\beta}_2\right) = \sigma_1^2 \text{plim}\left(\frac{X_1'X_1}{n_1}\right)^{-1} + \sigma_2^2 \text{plim}\left(\frac{X_2'X_2}{n_2}\right)^{-1}.$$

- The second equality uses the independence assumption.

Model

$$
\begin{aligned}
y &= \begin{pmatrix} Z_1 \\ \vdots \\ Z_{m+1} \end{pmatrix} \gamma + \begin{pmatrix} X_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & X_{m+1} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{m+1} \end{pmatrix} + \varepsilon \\
&= Z\gamma + X\beta + \varepsilon
\end{aligned}
$$

where Z_i and X_i are $T_i \times p$ and $T_i \times q$ matrices, respectively, and $T = \sum_{i=1}^{m+1} T_i$.

Let $\hat{\gamma}$ and $\hat{\beta}_j$ be the OLS estimators of γ and β_j, respectively, obtained by minimizing

$$\sum_{i=1}^{m+1} \sum_{t=T_{i-1}+1}^{T_i} (y_t - z_t'\gamma - x_t'\beta_i)^2.$$

Substituting $\hat{\gamma}$ and $\hat{\beta}_j$ in the objective function and denote the resulting sum of squared residuals as $S_T(T_1, ..., T_m)$.Then, the estimated change dates are

$$(\hat{T}_1, ..., \hat{T}_m) = \arg \min_{T_1,...,T_m} S_T(T_1, ..., T_m),$$

where the minimization is taken over all partitions $(T_1, ..., T_m)$ such that $T_i - T_{i-1} \geq q$.
Let $T_i = [T\lambda_i]$ where $0 < \lambda_1 < ... < \lambda_m < 1$. Then, as $T_i \to \infty$ for all i,

$$\hat{\lambda}_i \xrightarrow{p} \lambda_i \text{ for } i = 1, ..., m.$$

See Bai and Perron (1998, Econometrica) for assumptions and proofs.

Test the null hypothesis $H_0 : m = 0$ against the alternative $H_1 : m = k$. Consider the Wald test statistic given $\lambda_1, ..., \lambda_k$

$$W(\lambda_1, ..., \lambda_k) = \frac{\hat{\beta}' R' \left(R(X' M_Z X) R' \right)^{-1} R\hat{\beta}}{SSR_k / (T - (k+1)q - p)},$$

where $\left(R\hat{\beta}\right)' = (\hat{\beta}_1' - \hat{\beta}_2', ..., \hat{\beta}_k' - \hat{\beta}_{k+1}')$ and SSR_k is the sum of squared residuals under the alternative.

Test statistic for unknown $\lambda_1, ..., \lambda_k$

$$\sup_{(\lambda_1,...,\lambda_k)\in\Lambda} W(\lambda_1, ..., \lambda_k).$$

See Table 1 of Bai and Perron (1998) for critical values of the sup-Wald test.