Solutions to Midterm Examination

Advanced Econometrics (Spring, 2010)

1. Discuss the validity of the following statements. Simple "Yes" or "No" answer will not be accepted for credit.

   (a) (5 points) When regressors are uncorrelated among themselves, we can obtain more efficient estimates of slope coefficients than when they are not.
   Ans. (True) Consider the variance of $\hat{\beta}_k$ in multiple regressions

   $$Var(\hat{\beta}_k|X) = \frac{\sigma^2}{\sum(x_{ik} - \bar{x}_k)^2(1 - R_k^2)}.$$

   If regressors are less correlated among themselves, R_k^2 will become close to zero and we have more efficient estimates of slope coefficients.

   (b) (5 points) When more regressors are added, the MLE of the variance of error terms takes larger values.
   Ans. (False)

   $$\hat{\sigma}^2_{MLE} = \frac{(y - X\hat{\beta}_{MLE})'(y - X\hat{\beta}_{MLE})}{n} = \frac{e'e}{n}$$

   Since the numerator will always decrease when more regressors are added, $\hat{\sigma}^2_{MLE}$ will take smaller values.

   (c) (5 points) The null hypothesis $H_0 : R^2 = 0$ can be tested by using the F-test.
   Ans. (True) We can test the null hypothesis $H_0 : R^2 = 0$ by using the relation

   $$F = \frac{R^2/(K-1)}{(1-R^2)/(n-K)}.$$

   (d) (5 points) The MLE of the error variance in the linear regression model minimizes the mean squared error.
   Ans. (False) The mean squared error is minimized by $\hat{\sigma}^2 = \frac{e'e}{n-K+2}$.

   (e) (5 points) The OLS estimator for the ARMA(1,1) model with its AR coefficient less than one in modulus is consistent.
   Ans. (False) By repeated substitutions, we obtain

   $$y_t = \varepsilon_t + (\theta + \alpha)\varepsilon_{t-1} + \alpha(\theta + \alpha)\varepsilon_{t-2} + ...$$

   which shows that y_{t-1} and ε_{t-1} are correlated. Therefore, the OLS estimator is not consistent.

   (f) (5 points) If the null hypothesis of the White test for heteroskedasticity is rejected, we need to use OLS without any adjustments.
   Ans. (False) The null hypothesis of White's test for heteroskedasticity is no heteroskedasticity. Therefore, when it is rejected, White's adjustment of standard errors should be used along with the OLS estimator in order to test for a null hypothesis on coefficients.

(g) (5 points) Wald test for a linear restriction on the coefficients of a linear regression model is more power than the corresponding likelihood ratio and Lagrange multiplier tests.

Ans. (False) Since the Wald test statistic has a larger value than the others both under the null and alternative, its seemingly higher power comes at the cost of higher-than-specified Type-I error. Thus, the size-corrected power of the Wald test is no different from those of the likelihood ratio and Lagrange multiplier tests.

(h) (5 points) Biased estimators are often used in practice, but inconsistent estimators are never used.

Ans. (True) An example of a biased estimator that is often used in practice is the OLS estimator for AR models. However, if an estimator is inconsistent, it is not recommended for use.

2. (20 points) Suppose that z_i (a scalar variable) is a valid instrument for x_i in the model

$$y_i = \mu + \beta x_i + \varepsilon_i.$$

One considers the following OLS regression

$$\hat{y}_i = \hat{\mu} + \hat{\beta} x_i + \hat{\gamma} w_i,$$

where w_i is the OLS regression residual obtained by regressing x_i on z_i. How is $\hat{\beta}$ related to the usual IV estimator that uses $(1, z_i)$ as an instrument?

Ans. The OLS regression can be written in a deviation form as

$$\begin{aligned} \hat{y}_i - \bar{y} &= \hat{\beta}(x_i - \bar{x}) + \hat{\gamma}(w_i - \bar{w}) \\ &= \hat{\beta} x_i^* + \hat{\gamma} w_i^*. \end{aligned}$$

Thus, we may write

$$\hat{\beta} = (x^{*\prime}(I - P_{w^*})x^*)^{-1} x^{*\prime}(I - P_{w^*})y^*,$$

where $y^* = [y_1 - \bar{y}, ..., y_n - \bar{y}]'$. When x^* is regressed on $w^* = (I - P_z)x^*$, the OLS coefficient estimator is 1, implying that the residual is $x^* - w^* = P_z x^*$. Thus,

$$(I - P_{w^*})x^* = P_z x^*$$

and

$$\hat{\beta} = (x^{*\prime} P_z x^*)^{-1} x^{*\prime} P_z y^*.$$

3. (20 points) The true data generating process (DGP) is $y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i$, where $\{\varepsilon_i\}$ satisfy standard assumptions of the classical linear regression model. However, the OLS regression $\hat{y}_i = \hat{\beta} x_{1i}$ was run.

(a) Is the OLS estimator of β unbiased?

Ans. The OLS estimator in the underspecified model is

$$\begin{aligned} \hat{\beta} &= \frac{\sum x_{1i} y_i}{\sum x_{1i}^2} \\ &= \frac{\sum x_{1i}(\beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i)}{\sum x_{1i}^2} \\ &= \beta_1 + \beta_2 \frac{\sum x_{1i} x_{2i}}{\sum x_{1i}^2} + \frac{\sum x_{1i} \varepsilon_i}{\sum x_{1i}^2} \end{aligned}$$

Taking expectations on both sides yields

$$E\hat{\beta} = \beta_1 + \beta_2 \frac{\sum x_{1i}x_{2i}}{\sum x_{1i}^2}.$$

Therefore, the OLS estimator of β is biased, unless $\sum x_{1i}x_{2i} = 0$.

(b) Is the OLS estimator of β consistent?
Ans. From (a) we have

$$\hat{\beta} = \beta_1 + \beta_2 \frac{\sum x_{1i}x_{2i}}{\sum x_{1i}^2} + \frac{\sum x_{1i}\varepsilon_i}{\sum x_{1i}^2}$$

Assuming $\frac{1}{n}\sum x_{1i}x_{2i} \xrightarrow{p} m_{12}(\neq 0)$, $\frac{1}{n}\sum x_{1i}^2 \xrightarrow{p} m_{11}(> 0)$ and $\frac{1}{n}\sum x_{1i}\varepsilon_i \xrightarrow{p} 0$, we obtain

$$\hat{\beta} \xrightarrow{p} \beta_1 + \beta_2 m_{11}^{-1} m_{12} \neq \beta_1$$

Thus, the OLS estimator of β is inconsistent, unless $m_{12} = 0$.

4. (20 point) In a regression model

$$y_i = \alpha + \beta x_i + \varepsilon_i,$$

the regressor and errors are correlated. But a valid instrument z_i is available. Consider switching the independent and dependent variables and running an IV regression using z_i as an instrument. Is the IV estimator consistent? What is its limiting distribution?
Ans. The true model after switching the independent and dependent variables is written as

$$\begin{aligned} x_i &= -\frac{\alpha}{\beta} + \frac{1}{\beta} y_i - \frac{1}{\beta}\varepsilon_i \\ &= \gamma + \omega y_i + e_i. \end{aligned}$$

Write the IV estimator as

$$\begin{aligned} \hat{\omega}_{IV} &= \frac{\sum_{i=1}^n (z_i - \bar{z}) x_i}{\sum_{i=1}^n (z_i - \bar{z})(y_i - \bar{y})} \\ &= \frac{\sum_{i=1}^n (z_i - \bar{z})\left(-\frac{\alpha}{\beta} + \frac{1}{\beta} y_i - \frac{1}{\beta}\varepsilon_i\right)}{\sum_{i=1}^n (z_i - \bar{z})(y_i - \bar{y})} \\ &= \frac{1}{\beta} - \frac{1}{\beta} \frac{\sum_{i=1}^n (z_i - \bar{z})\varepsilon_i}{\sum_{i=1}^n (z_i - \bar{z})(y_i - \bar{y})}. \end{aligned}$$

Since

$$\frac{1}{n}\sum_{i=1}^n (z_i - \bar{z})\varepsilon_i \xrightarrow{p} 0$$

and

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^n (z_i - \bar{z})(y_i - \bar{y}) &= \frac{1}{n}\sum_{i=1}^n \left[(z_i - \bar{z})(\beta(x_i - \bar{x}) + (\varepsilon_i - \bar{\varepsilon}))\right] \\ &= \frac{1}{n}\beta \sum_{i=1}^n (z_i - \bar{z})(x_i - \bar{x}) + \frac{1}{n}\sum_{i=1}^n (z_i - \bar{z})(\varepsilon_i - \bar{\varepsilon}) \\ &\xrightarrow{p} \beta Cov(z, x) \end{aligned}$$

we obtain

$$\hat{\omega}_{IV} = \frac{1}{\beta} - \frac{1}{\beta}\frac{\frac{1}{n}\sum_{i=1}^{n}(z_i - \bar{z})\varepsilon_i}{\frac{1}{n}\sum_{i=1}^{n}(z_i - \bar{z})(y_i - \bar{y})} \qquad (1)$$

$$\xrightarrow{p} \frac{1}{\beta}$$

which shows that the IV estimator is consistent.

In addition, since the numerator of the second term in (1) follows a normal distribution asymptotically as

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(z_i - \bar{z})\varepsilon_i \xrightarrow{d} N(0, \sigma_z^2\sigma_\varepsilon^2),$$

we obtain

$$\sqrt{n}\left(\hat{\omega}_{IV} - \frac{1}{\beta}\right) = \frac{1}{\beta}\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(z_i - \bar{z})\varepsilon_i}{\frac{1}{n}\sum_{i=1}^{n}(z_i - \bar{z})(y_i - \bar{y})}$$

$$= \frac{1}{\beta}\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(z_i - \bar{z})\varepsilon_i}{\frac{1}{n}\sum_{i=1}^{n}(z_i - \bar{z})(y_i - \bar{y})}$$

$$\xrightarrow{d} N(0, \frac{\sigma_z^2\sigma_\varepsilon^2}{\beta^4 Cov(z, x)^2}).$$