Introduction to Econometrics
Sogang University

In Choi
Spring 2017

Home Assignment 4

Due: June 8, 2017

1. Consider the panel data model

$$y_{it} = \mu + \beta x_{it} + u_{it}, (i = 1, ..., N; t = 1, ..., T) \tag{1}$$

where $x_{it} \sim iid(0, \sigma_x^2)$, $u_{it} = \mu_i + v_{it}$, $v_{it} \sim iid(0, \sigma_v^2)$, and x_{it} is independent of v_{js} for all i, t, j and s.

(a) Assuming that $N \to \infty$ and that T is fixed, calculate the variance of $\sqrt{N}(\hat{\beta}_{FE} - \beta)$, where $\hat{\beta}_{FE}$ is the fixed effect estimator of β.
Ans. The FE estimator is written as

$$\hat{\beta}_{FE} - \beta = \frac{\sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_{i.}) v_{it}}{\sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_{i.})^2}.$$

Since $E\left(\sum_{t=1}^{T} (x_{it} - \bar{x}_{i.})^2\right) = (T-1)\sigma_x^2$, the law of large numbers gives

$$\frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_{i.})^2 \xrightarrow{p} (T-1)\sigma_x^2.$$

In addition, $\sum_{t=1}^{T} (x_{it} - \bar{x}_{i.}) v_{it} \sim (0, (T-1)\sigma_x^2 \sigma_v^2)$ for each T, the CLT gives

$$\begin{aligned} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_{i.}) v_{it} &= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \sum_{t=1}^{T} x_{it} v_{it} + o_p(1) \\ &\xrightarrow{d} N\left(0, (T-1)\sigma_x^2 \sigma_v^2\right). \end{aligned}$$

Thus,

$$\sqrt{N}\left(\hat{\beta}_{FE} - \beta\right) \xrightarrow{d} N\left(0, \frac{\sigma_v^2}{(T-1)\sigma_x^2}\right).$$

(b) Consider the differenced model

$$\Delta y_{it} = \beta \Delta x_{it} + \Delta v_{it}.$$

Under the same assumptions as in part (a), calculate the variance $\sqrt{N}(\hat{\beta}_{OLS}-\beta)$, where $\hat{\beta}_{OLS}$ is the OLS estimator of β using the differenced model.
Ans. Write

$$\hat{\beta}_{OLS} = \beta + \frac{\sum_{i=1}^{N}\sum_{t=2}^{T}\Delta x_{it}\Delta v_{it}}{\sum_{i=1}^{N}\sum_{t=2}^{T}x_{it}^2}$$

or

$$\sqrt{N}\left(\hat{\beta}_{OLS}-\beta\right) = \frac{\sum_{i=1}^{N}\sum_{t=2}^{T}\Delta x_{it}\Delta v_{it}/\sqrt{N}}{\sum_{i=1}^{N}\sum_{t=2}^{T}\Delta x_{it}^2/N}.$$

Since

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{t=2}^{T}\Delta x_{it}^2 \xrightarrow{d} 2(T-1)\sigma_x^2$$

and

$$\begin{aligned} &Var\left(\sum_{i=1}^{N}\sum_{t=2}^{T}\Delta x_{it}\Delta v_{it}/\sqrt{N}\right) \\ = \; &(6T-8)\,\sigma_x^2\sigma_v^2, \end{aligned}$$

the asymptotic variance is

$$\begin{aligned} \frac{\sigma_v^2}{\sigma_x^2}\left[\frac{(6T-8)}{4(T-1)^2}\right] &= \frac{\sigma_v^2}{(T-1)\sigma_x^2}\left[\frac{(6T-8)}{4(T-1)}\right] \\ &= Asy.Var\left(\sqrt{N}\left(\hat{\beta}_{FE}-\beta\right)\right)\times\frac{(6T-8)}{4(T-1)}. \end{aligned}$$

(c) Which estimator is more efficient?
Ans. Because

$$\begin{aligned} &\frac{(6T-8)}{4(T-1)} \\ = \; &\begin{cases} =1, & T=2 \\ >1, & T>2 \end{cases}, \end{aligned}$$

the two estimators are asymptotically equivalent when $T=2$ and the FE estimator is more efficient when $T>2$.

2. Consider the simultaneous equation model

$$y_{1t} = \beta y_{2t} + \alpha z_{1t} + u_t. \quad (2)$$

The reduced form model for this model is written as

$$\begin{aligned} y_{1t} &= \pi_{11}z_{1t} + \pi_{12}z_{2t} + v_{1t}; \\ y_{2t} &= \pi_{21}z_{1t} + \pi_{22}z_{2t} + v_{2t}, \end{aligned}$$

where $(v_{1t}, v_{2t})' \sim iid\ (0, \Sigma)$.[1] Derive the probability limit of the OLS estimator of β using the structural equation (2). Is it consistent?
Ans. Let $x = (x_1, ..., x_T)'$, $P_x = x(x'x)^{-1}x'$ $(x = y_1, y_2, z_1, z_2, u, v_1, v_2)$, $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$ and $Z = \begin{pmatrix} z_1 & z_2 \end{pmatrix}$. Write

$$\begin{aligned} \hat{\beta} &= (y_2'(I - p_{z_1})y_2)^{-1} y_2'(I - p_{z_1})y_1 \\ &= \beta + (y_2'(I - p_{z_1})y_2)^{-1} y_2'(I - p_{z_1})u. \end{aligned}$$

As usual, assume $\frac{1}{n} z_i' v_j \xrightarrow{p} 0$ $(i, j = 1, 2)$ and $\frac{1}{n} Z'Z \xrightarrow{p} \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} = M$ (a positive definite matrix). These assumptions and the WLLN provide

$$\begin{aligned} & \frac{1}{n} y_2'(I - p_{z_1})y_2 \\ = \; & \pi_{22}^2 \frac{1}{n} z_2'(I - P_{z_1})z_2 + \frac{1}{n} v_2'(I - P_{z_1})v_2 \\ & \xrightarrow{p} \pi_{22}^2 \left(m_{22} - m_{12}^2/m_{11}\right) + \sigma_{11}, \end{aligned} \tag{3}$$

as $n \to \infty$,where n is the sample size. Since $u_t = v_{1t} - \beta v_{2t}$, we obtain

$$\begin{aligned} & \frac{1}{n} y_2'(I - p_{z_1})u \\ = \; & \pi_{22} \frac{1}{n} z_2'(I - P_{z_1})u + \frac{1}{n} v_2'(I - P_{z_1})u \\ & \xrightarrow{p} -\beta\sigma_{22}, \end{aligned} \tag{4}$$

as $n \to \infty$. Relations (3) and (4) yield

$$\hat{\beta} \xrightarrow{p} \beta - \left[\pi_{22}^2 \left(m_{22} - m_{12}^2/m_{11}\right)\right]^{-1} \beta\sigma_{22}.$$

Thus, the OLS estimator is inconsistent.

3. Let the true regression model be

$$y_i = \alpha + \beta x_i + \varepsilon_i.$$

But we observe

$$x_i^* = x_i + w_i \;\; \left(w_i \sim iid\left(0, \sigma_w^2\right)\right)$$

instead of x_i due to measurement error. When y_i is regressed on x_i^*, what is the probability limit of the OLS estimator? Assume that $\{\varepsilon_i\}$ and $\{w_i\}$ are

[1] $\Sigma = \begin{pmatrix} Ev_{1t}^2 & Ev_{1t}v_{2t} \\ Ev_{1t}v_{2t} & Ev_{2t}^2 \end{pmatrix}$ for all t.

independent.
Ans. The regression model we use is

$$\begin{aligned} y_i &= \alpha + \beta\left(x_i^* - w_i\right) + \varepsilon_i \\ &= \alpha + \beta x_i^* + \varepsilon_i - \beta w_i. \end{aligned}$$

The OLS estimator of β is written as

$$\begin{aligned} \hat{\beta} &= \beta + \frac{\sum_{i=1}^n (x_i^* - \bar{x}^*)(\varepsilon_i - \beta w_i)}{\sum_{i=1}^n (x_i^* - \bar{x}^*)^2} \\ &= \frac{\sum_{i=1}^n (x_i - \bar{x} + w_i - \bar{w})(\varepsilon_i - \beta w_i)}{\sum_{i=1}^n (x_i - \bar{x} + w_i - \bar{w})^2}. \end{aligned}$$

Assume, as usual, $\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2 \xrightarrow{p} m$ and $\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})\varepsilon_i \xrightarrow{p} 0$. In addition, assume $\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x}) w_i \xrightarrow{p} l$. Since

$$\begin{aligned} & \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x} + w_i - \bar{w})^2 \\ & \xrightarrow{p} m + \sigma_w^2 + 2l \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x} + w_i - \bar{w})(\varepsilon_i - \beta w_i) \\ & \xrightarrow{p} -\beta\sigma_w^2, \end{aligned}$$

due to the assumptions and the WLLN, we have

$$\hat{\beta} \xrightarrow{p} \beta - \frac{\beta\sigma_w^2}{m + \sigma_w^2 + 2l},$$

proving inconsistency of the OLS estimator.