Proof of the Gauss-Markov Theorem

The case of $k=1$.

Consider a linear estimator

$$\tilde{\beta}_1 = \sum_{t=1}^{T} a_t Y_t$$

$$= \beta_0 \sum_t a_t + \beta_1 \sum a_t X_t + \sum a_t u_t .$$

For $\tilde{\beta}_1$ to be unbiased, we need to have

$$\sum a_t = 0 \quad \& \quad \sum a_t X_t = 1 .$$

The variance of $\tilde{\beta}_1$ is

$$Var(\tilde{\beta}_1) = \sigma_u^2 \sum a_t^2 .$$

Let $a_t = w_t + d_t$ where $w_t = \dfrac{X_t - \bar{X}}{\sum (X_t - \bar{X})^2}$.

Then,

$$\sum a_t^2 = \sum w_t^2 + 2 \sum w_t d_t + \sum d_t^2 .$$

But

$$\sum w_t d_t = \frac{\sum (X_t - \bar{X}) d_t}{\sum (X_t - \bar{X})^2}$$

$$= \frac{\sum X_t d_t - \bar{X} \sum d_t}{\sum (X_t - \bar{X})^2}$$

$$= \frac{\Sigma x_t (a_t - w_t) - \bar{x}\,\Sigma (a_t - w_t)}{\Sigma (x_t - \bar{x})^2}$$

$$= \frac{1 - 1 - \bar{x}(0 - 0)}{\Sigma (x_t - \bar{x})^2}$$

$$= 0 \quad ,$$

which gives

$$\Sigma a_t^2 = \Sigma w_t^2 + \Sigma d_t^2$$

$$\geq \Sigma w_t^2$$

$$= \cancel{\operatorname{Var}(\hat{\beta}_{1,OLS}) / \sigma_u^2} .$$

Thus

$$\operatorname{Var}(\tilde{\beta}_1) = \sigma_u^2\, \Sigma a_t^2 \geq \sigma_u^2\, \Sigma w_t^2$$

$$= \operatorname{Var}(\hat{\beta}_{1,OLS}) .$$