Econometrics

Method-of-Moments and Maximum Likelihood Estimators

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Method of Moments Estimators

- Let $X_1, ..., X_n$ be an iid sample from a population with pdf $f(x \mid \theta_1, ..., \theta_k)$.
- Let $\mu_j' = EX^j$ $(j = 1, ..., k)$. The population moments are typically a function of $(\theta_1, ..., \theta_k)$.
- Define

$$m_j = \frac{1}{n} \sum_{i=1}^{n} X_i^j, \ (j = 1, ..., k).$$

The MM estimator is obtained by the following system of equations

$$m_j = \mu_j'(\theta_1, ..., \theta_k), \ (j = 1, ..., k).$$

Method of Moments Estimators

Example

Suppose $X_i \sim iid\ N(\theta, \sigma^2)$ $(i = 1, ..., n)$. We have $m_1 = \bar{X}$ and $m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$, $\mu_1' = \theta$ and $\mu_2' = \theta^2 + \sigma^2$. The MM estimators are obtained by solving the equations

$$
\begin{aligned}
\bar{X} &= \theta; \\
\frac{1}{n}\sum_{i=1}^{n} X_i^2 &= \theta^2 + \sigma^2.
\end{aligned}
$$

Thus $\hat{\theta} = \bar{X}$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

Method of Moments Estimators

Example

Suppose that $\{X_t, t = 1, ..., n\}$ is a stationary process. Then,

$$\begin{aligned} EX_t &= m \\ E(X_t - m)(X_{t+h} - m) &= EX_tX_{t+h} - m^2 = \gamma(h). \end{aligned}$$

The MM estimators of m and $\gamma(h)$ are obtained by solving the equations

$$\begin{aligned} \bar{X} &= m \\ n^{-1}\sum_{t=1}^{n-h}(X_t - m)(X_{t+h} - m) &= \gamma(h). \end{aligned}$$

Maximum Likelihood Estimators

- Let $X_1, \ldots, X_n$ be an iid sample from a population with pdf $f(x \mid \theta_1, \ldots, \theta_k)$. The likelihood function si defined by

$$
\begin{aligned}
L(\theta \quad \mid \quad \mathbf{x}) &= L(\theta_1, \ldots, \theta_k \mid x_1, \ldots, x_n) \\
&= \Pi_{i=1}^{n} f(x_i \mid \theta_1, \ldots, \theta_k).
\end{aligned}
$$

- For each sample point $\mathbf{x}$, let $\hat{\theta}(\mathbf{x})$ be a parameter value at which $L(\theta \mid \mathbf{x})$ attains its maximum as a function of θ, with $\mathbf{x}$ held fixed. A maximum likelihood estimator (MLE) of the parameter θ based on a sample $\mathbf{x}$ is $\hat{\theta}(\mathbf{x})$.

Maximum Likelihood Estimators

- The MLE is the parameter point for which the observed sample is most likely.
- The MLE is a good point estimator, possessing some of the optimality properties.

Maximum Likelihood Estimators

Example

Suppose $X_i \sim iid\ N(\theta, 1)$ $(i = 1, ..., n)$. Then,

$$L(\theta \mid \mathbf{x}) = \Pi_{i=1}^{n} \frac{1}{(2\pi)^{1/2}} e^{-(x_i-\theta)^2/2} = \frac{1}{(2\pi)^{n/2}} e^{-\sum_{i=1}^{n}(x_i-\theta)^2/2}.$$

The first-order condition gives

$$\sum_{i=1}^{n}(x_i - \theta) = 0,$$

giving $\hat{\theta} = \bar{x}$. The second-order conditon $(\frac{\partial^2}{\partial\theta^2} L(\theta \mid \mathbf{x})_{|\theta=\bar{X}} < 0)$ is also satisfied. Thus, $\bar{x}$ is the only extreme point in the interior and it yields a maximum. At the boundaries $\theta = \pm\infty$, $L(\theta \mid \mathbf{x}) = 0$. Thus, $\bar{x}$ yields the global maximum likelihood.

Maximum Likelihood Estimators

Example

(Bernoulli MLE) $X_i \sim iid\ B(1, p)$ $(i = 1, ..., n)$. The likelihood function is

$$L(p \mid \mathbf{x}) = \Pi_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^y(1-p)^{n-y},$$

where $y = \sum_{i=1}^{n} x_i$. The log likelihood function is

$$\log L(p \mid \mathbf{x}) = y \log p + (n-y)\log(1-p).$$

If $0 < y < n$, $\frac{\partial}{\partial p} L(p \mid \mathbf{x}) = 0$ gives $\hat{p} = y/n$. If $y = 0$ or $y = n$,

$$\log L(p \mid \mathbf{x}) = \begin{matrix} n\log(1-p) & \text{if } y = 0 \\ n \log p & \text{if } y = n \end{matrix}.$$

Because $n\log(1-p)$ and $n \log p$ are maximized at $p = 0$ and $p = 1$, $\hat{p} = y/n$ is the MLE in either case.