Introduction to Econometrics

Chapter 18: Testing for a Unit Root

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Autoregressive integrated moving average (ARIMA) model

- Popularized by Box and Jenkins (1976).
- If d is nonnegative integer, $\{X_t\}$ is an $ARIMA(p, d, q)$ process if

$$r_t = (1 - B)^d X_t$$

is an $ARMA(p, q)$ process.

Autoregressive integrated moving average (ARIMA) model

- Many economic time series are well represented by the $ARIMA(p, 1, q)$ model (See Nelson and Plosser, 1982, Journal of Monetary Economics). Examples are GNP,CPI,interest rate,exchange rate, etc.
- $\{r_t\}$ is said to have a stochastic trend. This is because $\{r_t\}$ does not show quickly fluctuating behavior.

- How do we know that $d = 1$? Perform unit root test.
- Consider the AR(1) model

$$r_t = \phi r_{t-1} + a_t, \ a_t \sim WN(0, \sigma^2).$$

Let

$$\widehat{\phi} = \sum_{t=2}^{T} r_t r_{t-1} / \sum_{t=2}^{T} r_{t-1}^2$$

When $|\phi| < 1$

$$\widehat{\phi} \simeq N(\phi, \frac{1 - \phi^2}{T})$$

or

$$\sqrt{T}(\widehat{\phi} - \phi) \xrightarrow{d} N(0, 1 - \phi^2)$$

for large T.

Autoregressive integrated moving average (ARIMA) model

- Thus,

$$t(\phi) = \frac{\widehat{\phi} - \phi}{\sqrt{\hat{\sigma}^2 \left(\sum r_{t-1}^2\right)^{-1}}} \xrightarrow{d} N(0,1),$$

where $\hat{\sigma}^2 = \frac{1}{T-1} \sum_{t=2}^{T} (r_t - \widehat{\phi} r_{t-1})^2$.

- However, when $\phi = 1$, $T(\widehat{\phi} - 1) \xrightarrow{d}$ a nonnormal distribution and

$$t(1) = \frac{\widehat{\phi} - 1}{\sqrt{\hat{\sigma}^2 (\sum r_{t-1}^2)^{-1}}} \xrightarrow{d} \text{ a nonnormal distribution}$$

The distribution of $T\left(\hat{\phi} - 1\right)$ and $t(1)$ are tabulated in Wayne Fuller's "Introduction to Statistical Time Series" (1976,Wiley). These are known as unit root tests. Critical values of these tests are taken from the LHS tails of the distributions.

- Alternatively, we may write the model as

$$\Delta r_t = \lambda r_{t-1} + a_t, \;\; a_t \sim WN(0, \sigma^2)$$

and test the null hypothesis $H_0 : \lambda = 0$. The test statistics are

$$T\hat{\lambda} \text{ and } \frac{\widehat{\lambda}}{\sqrt{\hat{\sigma}^2 (\sum r_{t-1}^2)^{-1}}}$$

- When

$$r_t - \mu = \phi(r_t - \mu) + u_t,$$

or

$$r_t = \mu(1-\phi) + \phi r_{t-1} + u_t,$$

$\widehat{\phi}$ also has a nonnormal distribution in the limit if $\phi = 1$. The unit root tests for this model are:

$$T(\widehat{\phi} - 1) \; (\widehat{\phi} = \frac{\sum_{t=2}^{T}(r_{t-1} - \bar{r}_-)(r_t - \bar{r})}{\sum_{t=2}^{T}(r_{t-1} - \bar{r}_-)^2})$$

$$\frac{\widehat{\phi} - 1}{\sqrt{\hat{\sigma}^2(\sum_{t=2}^{T}(r_{t-1} - \bar{r}_-)^2)^{-1}}}.$$

Autoregressive integrated moving average (ARIMA) model

- An AR(p) model

$$r_t = \phi_1 r_{t-1} + ... + \phi_p r_{t-p} + a_t, \;\; a_t \sim WN(0, \sigma^2)$$

can be written as

$$\Delta r_t = \lambda r_{t-1} + \sum_{j=2}^{p} w_j \Delta r_{t-j+1} + a_t, \;\; a_t \sim WN(0, \sigma^2)$$

where the values of $\lambda = \phi_1 + ... + \phi_p - 1$ and $w_j = -\sum_{k=j}^{p} \phi_k$. When there is a unit root, $\phi_1 + ... + \phi_p = 1$. Thus, the null of a unit root can be tested by testing $\lambda = 0$. The t-test for this null hypothesis is called the augmented Dickey-Fuller test. It has the same asymptotic distribution as the t-test for the AR(1) model.