

Econometrics

Chapter 9: Heteroskedasticity

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What is Heteroskedasticity?

- The assumption of homoskedasticity means that $E(u_t^2 \mid \text{all } x\text{'s}) = \sigma^2$ (σ^2 is a constant that does not change over i).
- If this is not true, that is if the variance of u_t is different for different values of the x 's, then the errors are heteroskedastic.

What is Heteroskedasticity?

Example

(Cross-section consumption function)

y_i : household i 's consumption

x_i : household i 's income

Consider the regression equation

$$y_i = \beta_0 + \beta_1 x_i + u_i.$$

Then, $\text{Var}(y_i | x_i) = \text{Var}(u_i | x_i)$. It is highly likely that $\text{Var}(y_i | x_i)$ is large when x_i is large.

Why Worry About Heteroskedasticity?

- OLS is still unbiased and consistent, even if we do not assume homoskedasticity.

Example

Consider the model

$$y_t = \beta_0 + \beta_1 x_t + u_t, \quad E(u_t) = 0 \text{ and } E(u_t^2) = \sigma_t^2.$$

Assume for simplicity that $\{x_t\}$ is a sequence of constants. Recall that

$$\hat{\beta}_1 - \beta_1 = \sum_{t=1}^n w_t u_t,$$

where $w_t = \frac{x_t - \bar{x}}{\sum_{t=1}^n (x_t - \bar{x})^2}$. Then, $E(\hat{\beta}_1) = \beta_1$ since

$$E(\hat{\beta}_1 - \beta_1) = E\left(\sum_{t=1}^n w_t u_t\right)$$

Why Worry About Heteroskedasticity?

- Furthermore, if we assume $\sigma_t^2 < M$ for all t ,

$$\begin{aligned} \text{Var}(\hat{\beta}_1) &= E \left[(\hat{\beta}_1 - \beta_1)^2 \right] \\ &= E \left[\left(\sum_{t=1}^n w_t u_t \right)^2 \right] \\ &= \sum_{t=1}^n w_t^2 (E u_t^2) \\ &= \sum_{t=1}^n w_t^2 \sigma_t^2 \\ &= \frac{\sum_{t=1}^n (x_t - \bar{x})^2 \sigma_t^2}{\left(\sum_{t=1}^n (x_t - \bar{x})^2 \right)^2} \leq \frac{M}{\sum_{t=1}^n (x_t - \bar{x})^2}. \end{aligned}$$

Thus, if $\sum_{t=1}^n (x_t - \bar{x})^2 \rightarrow \infty$ as n increases, the estimator is consistent.

Why Worry About Heteroskedasticity?

- The usual formula for standard errors should be different if we have heteroskedasticity. Thus, we can not use the usual t or F test for drawing inferences.

- Consider the simple linear regression model. Then, $\text{Var}(\hat{\beta}_1)$ can be estimated by

$$\frac{\sum_{t=1}^n (x_t - \bar{x})^2 \hat{u}_t^2}{(\sum_{t=1}^n (x_t - \bar{x})^2)^2}.$$

- Adjusted t-ratio is defined by

$$\frac{\hat{\beta}_1}{\sqrt{\frac{\sum_{t=1}^n (x_t - \bar{x})^2 \hat{u}_t^2}{(\sum_{t=1}^n (x_t - \bar{x})^2)^2}}}.$$

This has a standard normal distribution in the limit. If this is used, heteroskedasticity is taken care of.

- Write

$$\sqrt{n}(\hat{\beta}_1 - \beta_1) = \left(\frac{1}{n} \sum_{t=1}^n (x_t - \bar{x})^2 \right)^{-1} \frac{1}{\sqrt{n}} \sum_{t=1}^n (x_t - \bar{x}) u_t.$$

The variance of $\frac{1}{\sqrt{n}} \sum_{t=1}^n (x_t - \bar{x}) u_t$ is $\frac{1}{n} \sum_{t=1}^n (x_t - \bar{x})^2 \sigma_t^2$.

- By the WLLN,

$$\frac{1}{n} \sum_{t=1}^n (x_t - \bar{x})^2 u_t^2 - \frac{1}{n} \sum_{t=1}^n (x_t - \bar{x})^2 \sigma_t^2 \xrightarrow{p} 0.$$

Since $u_t^2 - \hat{u}_t^2 \xrightarrow{p} 0$, $\frac{1}{n} \sum_{t=1}^n (x_t - \bar{x})^2 \sigma_t^2$ can be estimated by $\frac{1}{n} \sum_{t=1}^n (x_t - \bar{x})^2 \hat{u}_t^2$.

- Thus, if $\lim \frac{1}{n} \sum_{t=1}^n (x_t - \bar{x})^2 = M$ and $\lim \frac{1}{n} \sum_{t=1}^n (x_t - \bar{x})^2 \sigma_t^2 = L$,

$$\frac{\hat{\beta}_1 - \beta_1}{\sqrt{\frac{\sum_{t=1}^n (x_t - \bar{x})^2 \hat{u}_t^2}{(\sum_{t=1}^n (x_t - \bar{x})^2)^2}}} = \frac{\sqrt{n} (\hat{\beta}_1 - \beta_1)}{\sqrt{\frac{\frac{1}{\sqrt{n}} \sum_{t=1}^n (x_t - \bar{x})^2 \hat{u}_t^2}{(\frac{1}{n} \sum_{t=1}^n (x_t - \bar{x})^2)^2}}} \xrightarrow{d} \frac{N(0, M^{-2}L)}{\sqrt{\frac{L}{M^2}}} = N(0, 1)$$

Robust Standard Errors

- Important to remember that these robust standard errors only have asymptotic justification. In small samples, the t-statistic formed with the robust standard error may not have a distribution close a standard normal distribution, and inferences may not not be correct.

Generalized least squares (GLS) estimation

- Model

$$y_t = \beta_0 + \beta_1 x_{1t} + \cdots + \beta_k x_{kt} + u_t.$$

If $\text{Var}(u_t) = \sigma_t^2$, consider the transformed model

$$y_t/\sigma_t = \beta_0/\sigma_t + \beta_1 x_{1t}/\sigma_t + \cdots + \beta_k x_{kt}/\sigma_t + u_t/\sigma_t.$$

- The error term u_t/σ_t satisfy classical assumptions. Thus, the OLS estimator from this transformed model is BLUE. This estimator is called the generalized least squares (GLS) estimator.
- In practice, σ_t is not known.