Optimal Autoregressive Predictions

In Choi*and Sun Ho Hwang†

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Abstract

This paper proposes a new, optimal estimator of the AR(1) coefficient that minimizes the prediction mean-squared-error. This estimator can be used to generate an optimal predictor. The new estimator's asymptotic distributions are derived for the cases of stationarity and a near unit root. The optimal estimator is also derived for the AR(p) model ($p \geq 2$) and its asymptotic distributions are reported. Simulation results confirm advantages of using the optimal estimator for prediction.

Keywords: Autoregressive model, prediction, near unit root

1 Introduction

Predicting time series is an important task in business and science for which various methods have been proposed (see Elliott, Granger and Timmerman, 2006, and Elliott and Timmerman, 2013 for comprehensive reviews). The most basic and popular method is to use the univariate autoregressive (AR) model. Although the univariate AR model is simple, more sophisticated models often fail to outperform this model. In this sense, the AR model is a bench-mark model that should be considered in every prediction exercise.

*Corresponding author. Professor, Department of Economics, Sogang University, Seoul, Korea. Personal Webpage: http://inchoi.sogang.ac.kr. E-mail: inchoi@gmail.com (preferred), inchoi@sogang.ac.kr.

†Graduate student, Department of Economics, Sogang University, Seoul, Korea. E-mail: sunho3137@sogang.ac.kr.

In using the AR(1) model $y_t = \alpha y_{t-1} + u_t$ for prediction, the predictor of y_{T+1} that minimizes the mean-squared-error is αy_T when α is known (cf. Brockwell and Davis, 2006). Because α is unknown, the common practice is replace α with the least squares estimator (LSE) of α, $\hat{\alpha}_{LSE}$. But one cannot say that $\hat{\alpha}_{LSE} y_T$ is an optimal predictor of y_{T+1} at fixed T, although it is so asymptotically.

This paper proposes a new, optimal estimator of the AR(1) coefficient that minimizes the prediction mean-squared-error. This estimator can be used to generate an optimal predictor. This paper's approach of using the new estimator is certainly different from the conventional one described above. The new estimator's asymptotic distributions are derived for the cases of stationarity and a near unit root. The optimal estimator is also derived for the AR(p) model ($p \geq 2$) and its asymptotic distributions are reported. Simulation results confirm advantages of using the optimal estimator for prediction.

This paper is planned as follows. Section 2 introduces the model, basic assumptions and estimator. Section 3 reports asymptotic distributions of the estimator introduced in Section 2. Section 4 extends the discussions so far to the AR(p) model. Section 5 reports simulation results. Section 6 provides a summary and further remarks. Proofs are relegated to appendices. All the limits are taken as $T \to \infty$. Weak convergence is denoted by $\rightsquigarrow$. $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real numbers and the set of positive real numbers, respectively.

2 The model, assumptions and estimator

Consider the AR(1) model

$$y_t = \alpha y_{t-1} + u_t, \ (t = 2, \ldots, T), \tag{1}$$

where $\{y_t\}$ denotes observed data and t is an index for time. Our main concern is to forecast y_{T+1} by using data from 2 to T.

Regarding the error term u_t, assume the following.

Assumption 1 *Assume that $\{u_t\}$ is a sequence of martingale differences with respect to the sequence of increasing σ-fields $\{\mathcal{F}_t\}$ such that* *(a)* $E(u_t^2 \mid \mathcal{F}_{t-1}) = \sigma_u^2$, $\sigma_u^2 \in \mathbb{R}_+$, *a.s. for all* t;

(b) $\sup_t E(|u_t|^{2+\varepsilon} \mid \mathcal{F}_{t-1}) < \infty$ *a.s. for some* $\varepsilon > 0$.

Suppose that we are concerned with predicting y_{T+1}. Denote the forecast of y_{T+1} as $\hat{y}_{T+1} = \alpha^* y_T$ where α^* is an unknown quantity. Then, the prediction mean-squared-error (PMSE) is

$$\begin{aligned} & E(\hat{y}_{T+1} - y_{T+1})^2 \\ = \; & E\left(\alpha^* y_T - (\alpha y_T + u_{T+1})\right)^2 \\ = \; & (\alpha^* - \alpha)^2 E y_T^2 + \sigma_u^2 - 2(\alpha^* - \alpha) E(y_T u_{T+1}) \\ = \; & (\alpha^* - \alpha)^2 E y_T^2 + \sigma_u^2, \end{aligned} \tag{2}$$

where the last equality is obtained because $E(y_T u_{T+1}) = EE(y_T u_{T+1} \mid \mathcal{F}_T) = E y_T E(u_{T+1} \mid \mathcal{F}_T) = 0$. Because $E y_T^2$ and σ_u^2 are unknown, replace them with y_T^2 and $\tilde{\sigma}_u^2 = \frac{1}{T-1} \sum_{t=2}^T (y_t - \alpha^* y_{t-1})^2$, respectively. An estimator of α^* is obtained by minimizing the resulting PMSE with respect to α^*. Letting $T_m = T - m$, the first order condition for the minimization problem,

$$(\alpha^* - \alpha) y_T^2 - \frac{1}{T_1} \sum_{t=2}^T y_t y_{t-1} + \alpha^* \frac{1}{T_1} \sum_{t=2}^T y_{t-1}^2 = 0,$$

yields an estimator of α^* as

$$\hat{\alpha}^* = \frac{\alpha y_T^2 + \frac{1}{T_1} \sum_{t=2}^T y_t y_{t-1}}{y_T^2 + \frac{1}{T_1} \sum_{t=2}^T y_{t-1}^2}.$$

This estimator will be called the prediction mean-squared-error minimizing estimator (PMME).

PMME is infeasible because it depends on the parameter α. In practice, we may replace α with any consistent estimator of α. A feasible version of $\hat{\alpha}_f^*$, called the feasible prediction mean-squared-error minimizing estimator (FPMME), is

$$\hat{\alpha}_f^* = \frac{\tilde{\alpha} y_T^2 + \frac{1}{T_1} \sum_{t=2}^T y_t y_{t-1}}{y_T^2 + \frac{1}{T_1} \sum_{t=2}^T y_{t-1}^2},$$

where $\tilde{\alpha}$ is a consistent estimator of α using data $y_2, ..., y_T$. Because

$$\hat{\alpha}_f^* = \left(\frac{y_T^2}{y_T^2 + \frac{1}{T_1} \sum_{t=2}^T y_{t-1}^2} \right) \tilde{\alpha} + \left(\frac{\frac{1}{T_1} \sum_{t=2}^T y_{t-1}^2}{y_T^2 + \frac{1}{T_1} \sum_{t=2}^T y_{t-1}^2} \right) \frac{\sum_{t=2}^T y_t y_{t-1}}{\sum_{t=2}^T y_{t-1}^2},$$

we find that FPMME is a weighted average of $\tilde{\alpha}$ and the LSE of α. Thus, if LSE is used for $\tilde{\alpha}$, FPMME ends up being LSE. In practice, we may use an M-estimator for $\tilde{\alpha}$. In Section 5, the least absolute deviation estimator (LADE) will be used for simulation.

The predictor of y_{T+h} using FPMME is $\hat{\alpha}_f^{*h} y_T$. This predictor is easy to use, but note that it is optimal only for $h = 1$. If we wish to predict y_{T+2}, for example, the same method as above yields the first order condition involving a polynomial equation of order 3, which does not admit a closed-form formula of the optimal predictor. In this case, numerical methods should be used to obtain a FPMME. This paper will focus only on the optimal prediction of y_{T+1}, relegating other cases to future work.

3 Asymptotic properties of PMME

Because PMME of the previous section is new in the literature, its properties need to be studied. This section reports asymptotic properties of the estimator $\hat{\alpha}^*$.

Asymptotic properties of the estimator $\hat{\alpha}^*$ are reported in the following theorem.

Theorem 1 *Suppose that Assumption 1 holds.*
(a) If $|\alpha| < 1$,

$$\sqrt{T_1}\,(\hat{\alpha}^* - \alpha) \rightsquigarrow \left(y_\infty^2 + \sigma_u^2/(1-\alpha^2)\right)^{-1} N(0, \frac{\sigma_u^4}{1-\alpha^2}).$$

(b) If $\alpha = \exp(\frac{c}{T})$, $c \in \mathbb{R}$ *and* $y_0 = O_p(1)$,

$$T_1\,(\hat{\alpha}^* - \alpha) \rightsquigarrow \left(J_c^2(1) + \int_0^1 J_c^2(r)\,\mathrm{d}r\right)^{-1} \int_0^1 J_c(r)\,\mathrm{d}W(r),$$

where $J_c(r) = \int_0^r e^{(r-s)c}\,\mathrm{d}W(s)$ *is the Ornstein–Uhlenbeck process and* $W(s)$ $(0 \le s \le 1)$ *is the standard Brownian motion.*

When $|\alpha| < 1$, Theorem 1 shows that PMME converges to α in probability at the rate of $\frac{1}{\sqrt{T}}$ and has a nonnormal limiting distribution. Its limiting distribution depends on those of the error terms. When $\alpha = \exp(\frac{c}{T})$, α is near to 1 because $\alpha = 1 + \frac{c}{T} + O(T^{-2})$. In the literature on time series analysis, $\{y_t\}$ is called the nearly integrated process. PMME in this case converges to α in probability at the rate of $\frac{1}{T}$ and has a nonnormal distribution in the limit. Its distribution is different from that of LSE reported in Chan and Wei (1987) and Phillips (1987). If the near unit roots are modelled in different ways, different limiting distributions will follow (see Chapter 2 of Choi, 2015). But analyzing different cases of near unit roots is beyond the scope of this paper and not pursued here.

4 Extension to the AR(p) model

This section extends the results of the previous sections to the AR(p) model ($p \geq 2$). The model we are concerned with is

$$\begin{aligned} y_t &= \alpha_1 y_{t-1} + ... + \alpha_p y_{t-p} + u_t \quad (t = p+1, \ldots, T) \\ &= \beta' X_t + u_t, \end{aligned}$$

where $\beta = (\alpha_1, ..., \alpha_p)'$ and $X_t = (y_{t-1}, ..., y_{t-p})'$. As in Section 2, we forecast y_{T+1} by using data from $p+1$ to T.

As in Section 2, the PMSE is

$$E(\hat{y}_{T+1} - y_{T+1})^2 = (\beta^* - \beta)' E X_T X_T' (\beta^* - \beta) + \sigma_u^2.$$

Replacing $EX_T X_T'$ and σ_u^2 with $X_T X_T'$ and $\tilde{\sigma}_u^2 = \frac{1}{T_p} \sum_{t=p+1}^{T} (y_t - \beta^{*\prime} X_t)^2$, respectively, the first order condition for the minimization of the PMSE is obtained as

$$X_T X_T' (\beta^* - \beta) - \frac{1}{T_p} \sum_{t=p+1}^{T} X_t y_t + \frac{1}{T_p} \sum_{t=p+1}^{T} X_t X_t' \beta^* = 0.$$

This yields an estimator of β^* as

$$\hat{\beta}^* = \left(X_T X_T' + \frac{1}{T_p} \sum_{t=p+1}^{T} X_t X_t' \right)^{-1} \left(X_T X_T' \beta + \frac{1}{T_p} \sum_{t=p+1}^{T} X_t y_t \right).$$

The feasible version of $\hat{\beta}^*$ is

$$\hat{\beta}_f^* = \left(X_T X_T' + \frac{1}{T_p} \sum_{t=p+1}^{T} X_t X_t' \right)^{-1} \left(X_T X_T' \tilde{\beta} + \frac{1}{T_p} \sum_{t=p+1}^{T} X_t y_t \right)$$

where $\tilde{\beta}$ is a consistent estimator of β. This estimator can be written as

$$\begin{aligned}\hat{\beta}^*_f &= \left(X_T X_T' + \frac{1}{T_p}\sum_{t=p+1}^{T} X_t X_t'\right)^{-1} X_T X_T' \tilde{\beta} \\ &+ \left(X_T X_T' + \frac{1}{T_p}\sum_{t=p+1}^{T} X_t X_t'\right)^{-1} \left(\frac{1}{T_p}\sum_{t=p+1}^{T} X_t X_t'\right) \hat{\beta}_{LSE},\end{aligned}$$

where $\hat{\beta}_{LSE}$ is the LSE of β. Thus, $\hat{\beta}^*_f$ is a linear combination of $\tilde{\beta}$ and $\hat{\beta}_{LSE}$.

Asymptotic properties of $\hat{\beta}^*$ depend on the roots of the characteristic equation

$$1 - \alpha_1 z - ... - \alpha_p z^p = 0. \tag{3}$$

Two cases are considered in this paper.

Case 1: All the roots of the characteristic equation (3) is outside the unit circle.

Case 2: $1 - \alpha_1 z - ... - \alpha_p z^p = (1 - \gamma z)(1 - \delta_1 z - ... - \delta_{p-1} z^{p-1}),\ \gamma = \exp(\frac{c}{T})$ $(c \in \mathbb{R})$, all the roots of the characteristic equation $1 - \delta_1 z - ... - \delta_{p-1} z^{p-1} = 0$ is outside the unit circle, and $p \geq 2$.

In Case 1, the time series $\{y_t\}$ is weakly stationary. In Case 2, $\{y_t - \exp(\frac{c}{T}) y_{t-1}\}$ is weakly stationary. Because most economic, financial and social time series become stationary after differencing once (see Chapter 1 of Choi, 2015), Case 2 is relevant to empirical time series analysis. Note that there is at least one stable root in Case 2.

Asymptotic properties of $\hat{\beta}^*$ are reported in the following theorem.

Theorem 2 *Suppose that Assumption 1 holds.*

(a) In Case 1,

$$\sqrt{T_p}(\hat{\beta}^* - \beta) \rightsquigarrow \left(X_\infty X_\infty' + \Gamma\right)^{-1} N\left(0, \sigma_u^2 \Gamma\right),$$

where $\Gamma = E\left(X_{p+1} X_{p+1}'\right)$.

(b) In Case 2, let $A_1 = \overbrace{\begin{bmatrix} 1 & & & \mathbf{0} \\ -\gamma & 1 & & \\ & -\gamma & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & -\gamma \end{bmatrix}}^{p-1} p$, $\mathcal{M}_{vv} = V_\infty V_\infty' + \Psi$ *with* $v_t = y_t -$

$\exp(\frac{c}{T})y_{t-1}$, $V_t = (v_{t-1}, ..., v_{t-p+1})'$ *and* $\Psi = E(V_{p+1}V'_{p+1})$, $\mathcal{M}_{vz} = \sigma_u V_\infty J_c(1)$, $\mathcal{M}_{zz} = \sigma_u^2 \left(J_c^2(1) + \int_0^1 J_c^2(r)\, \mathrm{d}r \right)$, $\mathcal{L} = \mathcal{M}_{zz} - \mathcal{M}'_{vz}\mathcal{M}_{vv}^{-1}\mathcal{M}_{vz}$ *and* $\sigma_v = 2\pi f_v(0)$ $(\sigma_v \in \mathbb{R}_+)$ *with* $f_v(\cdot)$ *denoting the spectral density of* $\{v_t\}$. *Then,*

$$\begin{aligned}\sqrt{T_p}(\hat{\beta}^* - \beta) \quad &\rightsquigarrow \quad A_1 \left(I_p + \mathcal{M}_{vv}^{-1}\mathcal{M}_{vz}\mathcal{L}^{-1}\mathcal{M}'_{vz}\right) \mathcal{M}_{vv}^{-1} N\left(0, \sigma_u^2 \Psi\right) \\ &\quad -A_1 \mathcal{M}_{vv}^{-1}\mathcal{M}_{vz}\mathcal{L}^{-1} \left(\sigma_u^2 \int_0^1 J_c(r)\mathrm{d}W(r) \right).\end{aligned}$$

In both the cases, the estimator $\hat{\beta}^*$ has nonnormal limiting distributions and $\sqrt{T_p}$-asymptotics holds. By contrast, we have

$$\sqrt{T_p}(\hat{\beta}_{LSE} - \beta) \rightsquigarrow N\left(0, \sigma_u^2 \Gamma^{-1}\right),$$

in Case 1, and

$$\sqrt{T_p}(\hat{\beta}_{LSE} - \beta) \rightsquigarrow N\left(0, \sigma_u^2 A_1 \Psi^{-1} A'_1\right).$$

in Case 2. See Choi (1993) for the latter result.

5 Simulation

This section reports simulation results regarding the performance of the predictors introduced in this paper. In addition, it contains simulation results comparing efficiency of PMME, FPMME, LADE and LSE.

5.1 Comparison of PMSEs

This subsection reports simulation results for empirical PMSEs of various predictors. For the AR(1) model, these employ LSE, PMME, FPMME using LADE, LADE, FPMME with $\alpha = 1$ and the random-walk predictor[1]. A predictor which is an average of those using LSE and LADE is also considered. This will be called the combination predictor here. For the AR(4) model, all the predictors except that using FPMME with $\alpha = 1$ and the random-walk predictor are considered.

[1] The random-walk predictor of y_{T+1} is y_T. Atkeson and Ohanian (2001) use this to predict inflation rates.

For the AR(1) model, data were generated by Model (1) where $\{u_t\}$ is a sequence of independent random variables that follow a standard normal or Student's t distribution. Degrees of freedom for Student's t distribution are 3 and 4.[2] We discarded the initial 30 observations to minimize the impact of the initial variable. The values of the AR(1) coefficient are set at 0.5, 0.80, 0.85, 0.90, 0.95, 0.96, 0.97, 0.98 and 0.99. Sample sizes 50, 100 and 200 are considered. The prediction horizons are 1, 3 and 5. The predictor using $\hat{\alpha}^*$, for example, is $\hat{\alpha}^{*h} y_T$, where h denotes the prediction horizon.

For the AR(4) model, data were generated by

$$(1 - \alpha_1 B)(1 - \alpha_2 B)(1 + 0.1B)(1 - 0.2B)y_t = u_t,$$

where B is the backward shift operator. The values of α_1 are set in the same manner as for the AR(1) model. For α_2, two values -0.10 and -0.75 are considered. The rest of the data generating scheme follow that of the AR(1) model discussed above.

Tables 1-3 contain empirical PMSEs for the AR(1) and AR(4) processes at $T =$ 50, 100 and 200. The number of iteration is 10,000. Three prediction horizons – 1, 3, and 5 – are considered. The PMSEs are standardized by that of LSE to make comparisons easier. The PMSEs less than 1 are interpreted to be better than that of LSE. The AR model without an intercept is used for prediction.

Table 1 contains results for the AR(1) process. The results are summarized as follows.

(i) The PMSE of PMME is smallest in every case we considered.

(ii) For the standard normal errors, the predictor using FPMME with $\alpha = 1$ performs better than all the feasible predictors including that using LSE when α is close to 1. At smaller T, the advantage of FPMME with $\alpha = 1$ stands out. But as T grows, the LSE-based predictor tends to perform better than that using FPMME with $\alpha = 1$.

(iii) When the errors follow Student's t distribution with 3 degrees of freedom, FPMME using LADE and FPMME with $\alpha = 1$ tend to dominate all the rest. When α

[2] Student's t-distribution with 6 degrees of freedom yields results similar to those of standard normal errors. These are not reported here to save the space.

is close to 1 and T is small, FPMME with $\alpha = 1$ shows better performance than FPMME using LADE, and vice versa. Notably, the predictor based on FPMME using LADE tend to show better performance than that using LADE and the combination predictor at every sample size.

(iv) When the errors follow Student's t distribution with 4 degrees of freedom, the predictor using FPMME with $\alpha = 1$ and the combination predictor tend to perform better that the rest. When α is close to 1 and T is small, the predictor using FPMME with $\alpha = 1$ shows better performance than the combination predictor, and vice versa.

Table 2 reports results for the AR(4) process with $\alpha_2 = -0.75$.[3] The results are summarized as follows.

(i) The PMSE of PMME is smallest in almost all the cases.

(ii) For the standard normal errors, LSE performs better than FPMME, the combination of LSE and LADE, and LADE at every sample size and prediction horizons considered.

(iii) When the errors follow Student's t distribution with 3 degrees of freedom, FPMME, the combination predictor and LADE perform better than LSE. In addition, FPMME performs better than the combination predictor and LADE at $h = 1$ without any exceptions. This reflects the fact that PMME is optimal for the case $h = 1$ as discussed in Section 2. At other prediction horizons, LADE tends to perform better than FPMME and the combination predictor at $T = 50,\ 100$. But at $T = 200$, FPMME tends to perform better than the combination predictor and LADE when $\alpha \geq 0.85$.

(iv) When the errors follow Student's t distribution with 4 degree of freedom, FPMME, the combination predictor and LADE tend to perform better than LSE. The combination predictor is shown to perform better than FPMME and LADE, although FPMME shows best performance in a few cases at $T = 50,\ 100$.

[3] We also have results for $\alpha_2 = -0.1$. These are qualitatively no different from those for $\alpha_2 = -0.75$, and are not reported here.

5.2 Efficiency and empirical pdf's of PMME and FPMME

This subsection reports efficiency comparisons of PMME, FPMME using LADE, LSE and LADE, and draws their empirical pdf's. In Table 3, empirical MSE, bias and variance of these estimators are reported for the AR(1) model with $\alpha = 0.5$ and $\alpha = 1 - \frac{2.5}{T}$. Sample sizes considered are 100 and 200. Error terms were generated as for Tables 1 and 2 using a standard normal or Student's t distribution with 3 degrees of freedom. Results in Table 3 can be summarized as follows.

(i) In every case, PMME has the least empirical MSE and variance.

(ii) The absolute value of the bias of PMME is smallest in most cases.

(ii) For the standard normal errors, LSE tends to have smaller MSE and variance than FPMME and LADE. But it tends to be more biased than FPMME and LADE. FPMME tends to rank in between LSE and LADE in relation to empirical MSE, bias and variance.

(iv) When the errors follow Student's t distribution with 3 degrees of freedom, LADE shows better performance than the rest in terms of empirical MSE, bias and variance. FPMME ranks in between LSE and LADE.

Empirical pdf's of PMME, FPMME using LADE, LSE and LADE are drawn in Figures 1 and 2. Figures 1 and 2 deal with the cases of stationary AR(1) model with $\alpha = 0.5$ and nearly integrated AR(1) model with $\alpha = 1 - \frac{2.5}{T}$. Figures 1 and 2 are summarized as follows.

(i) Figure 1 shows that PMME has almost symmetric pdf's and is highly peaked at the origin in every case. But in the case of a near unit root, its pdf is skewed to the left as shown in Figure 2.

(ii) FPMME mimics PMME reasonably well for in the case of stationarity and normally distributed errors. But it tends to be skewed to the left compared with PMME in other cases.

(iii) For the errors following Student's t distribution with 3 degrees of freedom, the empirical pdf's of LSE and LADE are quite skewed to the left when $\alpha = 1 - \frac{2.5}{T}$.

6 Summary and further remarks

We have considered the problem of how to generate optimal predictors using the AR model. In this paper, a new estimator of the AR(1) coefficient that minimizes the prediction mean-squared-error is derived. The estimator in turn is used to generate an optimal predictor. The new estimator's asymptotic distributions are derived for the cases of stationarity and a near unit root. The optimal estimator is also derived for the AR(p) model ($p \geq 2$) and its asymptotic distributions are reported. Simulation results show that the new approach of this paper has some merits.

7 Appendix I: Proofs

Proof of Theorem 1: (a) Write $\hat{\alpha}^* - \alpha = \left(y_T^2 + \frac{1}{T_1} \sum_{t=2}^{T} y_{t-1}^2 \right)^{-1} \left(\frac{1}{T_1} \sum_{t=2}^{T} y_{t-1} u_{t-1} \right)$. Then, standard theory of time series analysis (e.g., Theorem 2.2 of Chan and Wei, 1988) gives

$$\frac{1}{T_1} \sum_{t=2}^{T} y_{t-1}^2 \xrightarrow{p} \frac{\sigma_u^2}{1-\alpha^2} \tag{A.I.1}$$

and

$$\frac{1}{\sqrt{T_1}} \sum_{t=2}^{T} y_{t-1} u_t \rightsquigarrow N(0, \frac{\sigma_u^4}{1-\alpha^2}). \tag{A.I.2}$$

In addition, $E\left|y_T\right| \leq E \sum_{i=0}^{\infty} |\alpha|^i \left|u_{T-i}\right| = \sum_{i=0}^{\infty} |\alpha|^i E\left|u_{T-i}\right| < \infty$ for any T, where the equality holds by the monotone convergence theorem. This implies that y_T^2 is finite a.s. for any T. Thus, y_∞^2 is well defined. The Slutsky theorem using relations (A.I.1) and (A.I.2) proves the stated result.
(b) Recall that $T_1(\hat{\alpha}^* - \alpha) = \left(\frac{1}{T_1} y_T^2 + \frac{1}{T_1^2} \sum_{t=2}^{T} y_{t-1}^2 \right)^{-1} \left(\frac{1}{T_1} \sum_{t=2}^{T} y_{t-1} u_t \right)$. Because

$$\frac{1}{\sqrt{T_1}} y_T \rightsquigarrow \sigma_u J_c(1), \tag{A.I.3}$$

$$\frac{1}{T_1^2} \sum_{t=2}^{T} y_{t-1}^2 \rightsquigarrow \sigma_u^2 \int_0^1 J_c^2(r)\, \mathrm{d}r \tag{A.I.4}$$

$$\frac{1}{T_1} \sum_{t=2}^{T} y_{t-1} u_t \rightsquigarrow \sigma_u^2 \int_0^1 J_c(r)\, \mathrm{d}W(r) \tag{A.I.5}$$

jointly (see Phillips, 1987), the continuous mapping theorem yields the stated result.

□

Proof of Theorem 2: (a) Write

$$\sqrt{T_p}\left(\hat{\beta}^* - \beta\right) = \left(X_T X_T' + \frac{1}{T_p}\sum_{t=p+1}^{T} X_t X_t'\right)^{-1}\left(\frac{1}{\sqrt{T_p}}\sum_{t=p+1}^{T} X_t u_t\right).$$

The result follows because $\frac{1}{T_p}\sum_{t=p+1}^{T} X_t X_t' \xrightarrow{p} \Gamma$ and $\frac{1}{\sqrt{T_p}}\sum_{t=p+1}^{T} X_t u_t \rightsquigarrow N(0, \sigma_u^2\Gamma)$.

(b) Define a $p \times (p-1)$ band-diagonal matrix

$$A_1 = \begin{bmatrix} 1 & & & \mathbf{0} \\ -\gamma & 1 & & \\ & -\gamma & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & -\gamma \end{bmatrix}.$$

Let $V_t = A_1' X_t = (v_{t-1}, ..., v_{t-p+1})'$, where $v_t = y_t - \exp(\frac{c}{T}) y_{t-1}$. Note that $\{v_t\}$ follows an AR(p) process $v_t = \delta_1 v_{t-1} + ... + \delta_{p-1} v_{t-p+1} + u_t$ and is a weakly stationary process. In addition, let

$$A_2 = \begin{bmatrix} 1 \\ -\delta_1 \\ \vdots \\ -\delta_{p-1} \end{bmatrix}$$

and introduce $z_t = A_2' X_t$, where $z_t = \exp(\frac{c}{T}) z_{t-1} + u_t$. Note that $\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} z_t \rightsquigarrow \sigma_u J_c(r)$. Now, define $A = [A_1 \ A_2]$. The matrix A is nonsingular. Using a formula for the partitioned inverse (cf. Lütkepohl, 1996, p.147), write

$$\begin{aligned}
\hat{\beta}^* - \beta &= A\left(A'X_TX_T'A + \frac{1}{T_p}\sum_{t=p+1}^{T} A'X_tX_t'A\right)^{-1}\left(\frac{1}{T_p}\sum_{t=p+1}^{T} A'X_tu_t\right) \\
&= A\begin{pmatrix} A_1'X_TX_T'A_1 + \frac{1}{T_p}\sum_{t=p+1}^{T} A_1'X_tX_t'A_1 & A_1'X_TX_T'A_2 + \frac{1}{T_p}\sum_{t=p+1}^{T} A_1'X_tX_t'A_2 \\ A_2'X_TX_T'A_1 + \frac{1}{T_p}\sum_{t=p+1}^{T} A_2'X_tX_t'A_1 & A_2'X_TX_T'A_2 + \frac{1}{T_p}\sum_{t=p+1}^{T} A_2'X_tX_t'A_2 \end{pmatrix}^{-1} \\
&\quad \times \begin{pmatrix} \frac{1}{T_p}\sum_{t=p+1}^{T} A_1'X_tu_t \\ \frac{1}{T_p}\sum_{t=p+1}^{T} A_2'X_tu_t \end{pmatrix} \\
&= A_1B_1 + A_1B_2 + A_2B_3,
\end{aligned}$$

where

$$B_1 = \left(V_TV_T' + \frac{1}{T_p}\sum_{t=p+1}^{T} V_tV_t'\right)^{-1}\frac{1}{T_p}\sum_{t=p+1}^{T} V_tu_t,$$

$$\begin{aligned}
B_2 &= \left(V_TV_T' + \frac{1}{T_p}\sum_{t=p+1}^{T} V_tV_t'\right)^{-1}\left(V_Tz_T + \frac{1}{T_p}\sum_{t=p+1}^{T} V_tz_t'\right)\Lambda^{-1} \\
&\quad \times \left(z_TV_T' + \frac{1}{T_p}\sum_{t=p+1}^{T} z_tV_t'\right)\left(V_TV_T' + \frac{1}{T_p}\sum_{t=p+1}^{T} V_tV_t'\right)^{-1}\left(\frac{1}{T_p}\sum_{t=p+1}^{T} V_tu_t\right) \\
&\quad - \left(V_TV_T' + \frac{1}{T_p}\sum_{t=p+1}^{T} V_tV_t'\right)^{-1}\left(V_Tz_T + \frac{1}{T_p}\sum_{t=p+1}^{T} V_tz_t\right)\Lambda^{-1}\left(\frac{1}{T_p}\sum_{t=p+1}^{T} z_tu_t\right),
\end{aligned}$$

$$\begin{aligned}
B_3 &= -\Lambda^{-1}\left(z_TV_T' + \frac{1}{T_p}\sum_{t=p+1}^{T} z_tV_t'\right)\left(V_TV_T' + \frac{1}{T_p}\sum_{t=p+1}^{T} V_tV_t'\right)^{-1}\left(\frac{1}{T_p}\sum_{t=p+1}^{T} V_tu_t\right) \\
&\quad +\Lambda^{-1}\left(\frac{1}{T_p}\sum_{t=p+1}^{T} z_tu_t\right),
\end{aligned}$$

and

$$\begin{aligned}
\Lambda &= \left(z_T^2 + \frac{1}{T_p}\sum_{t=p+1}^{T} z_t^2\right) - \left(z_TV_T' + \frac{1}{T_p}\sum_{t=p+1}^{T} z_tV_t'\right) \\
&\quad \times \left(V_TV_T' + \frac{1}{T_p}\sum_{t=p+1}^{T} V_tV_t'\right)^{-1}\left(V_Tz_T + \frac{1}{T_p}\sum_{t=p+1}^{T} V_tz_t\right).
\end{aligned}$$

Lemma A.II.1 in Appendix II yields $B_1 = O_p(\frac{1}{\sqrt{T_p}})$, $B_2 = O_p(\frac{1}{\sqrt{T_p}})$ and $B_3 = O_p(\frac{1}{T_p})$. Thus, letting $M_{vv} = V_T V_T' + \frac{1}{T_p}\sum_{t=p+1}^{T} V_t V_t'$ and $M_{vz} = V_T z_T + \frac{1}{T_p}\sum_{t=p+1}^{T} V_t z_t$, we obtain

$$\begin{aligned}\sqrt{T_p}\left(\hat{\beta}^* - \beta\right) &= \left(A_1 + A_1 M_{vv}^{-1} M_{vz} \Lambda^{-1} M_{vz}'\right) M_{vv}^{-1} \frac{1}{\sqrt{T_p}} \sum_{t=p+1}^{T} V_t u_t \\ &\quad - A_1 M_{vv}^{-1} M_{vz} \Lambda^{-1} \frac{1}{T_p} \sum_{t=p+1}^{T} z_t u_t.\end{aligned}$$

The stated result follows from Lemma A.II.1. □

8 Appendix II: Auxiliary lemmas

Lemma A.II.1 *(a)* $V_T V_T' + \frac{1}{T_1}\sum_{t=2}^{T} V_t V_t' \rightsquigarrow V_\infty V_\infty' + \Psi$, *where* Ψ *is defined in Theorem 2.*

(b) $\frac{1}{\sqrt{T_1}}\left(V_T z_T + \frac{1}{T_1}\sum_{t=2}^{T} V_t z_t\right) \rightsquigarrow \sigma_u V_\infty J_c(1)$.

(c) $\frac{1}{T_1}\left(z_T^2 + \frac{1}{T_1}\sum_{t=2}^{T} z_t^2\right) \rightsquigarrow \sigma_u^2 J_c(1)^2 + \sigma_u^2 \int_0^1 J_c^2(r)\,\mathrm{d}r$.

(d) $\frac{1}{T_1}\sum_{t=2}^{T} V_t z_t \rightsquigarrow \sigma_u \int_0^1 J_c(r)\,\mathrm{d}B_V(r)$, *where* $B_V(r)$ *is the weak limit of* $\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} V_t$.

(e) $\frac{1}{T_1}\sum_{t=2}^{T} z_t^2 \rightsquigarrow \sigma_u^2 \int_0^1 J_c^2(r)\,\mathrm{d}r$

(f) $\frac{1}{\sqrt{T_1}}\sum_{t=2}^{T} V_t u_t \rightsquigarrow N\left(0, \sigma_u^2 \Psi\right)$.

(g) $\frac{1}{T_1}\sum_{t=2}^{T} z_t u_t \rightsquigarrow \sigma_u^2 \int_0^1 J_c(r)\,\mathrm{d}W(r)$.

Proof: Parts (a) and (f) are standard results in time series analysis. The rest are also standard in the literature on nonstationary time series analysis. See Phillips (1988). □

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Table 1: Empirical prediction mean-squared-errors: AR(1) model

Note: 1. Data were generated by $(1-\alpha B)y_t = u_t$, $u_t \sim i.i.d.N(0,1), t(3)$, or $t(4)$.
2. Entries of this table are empirical mean-squared-errors standardized by that of the LSE-based predictor.
3. FPMME1 is FPMME using LAD, and FPMME2 is FPMME using $\alpha = 1$.
4. The prediction horizon is denoted by h.
5. AVG denotes the mean of the prediction mean-squared-errors over the prediction horizons.
6. The number of Monte Carlo replications is 10,000.

Part A: $T = 50$

		$N(0,1)$						$t(3)$						$t(4)$					
α	h	PMME	FPMME1	FPMME2	RW	COMB	LADE	PMME	FPMME1	FPMME2	RW	COMB	LADE	PMME	FPMME1	FPMME2	RW	COMB	LADE
0.50	1	0.9821	1.0057	1.1362	1.3070	1.0025	1.0107	0.9627	0.9770	1.1775	1.3165	0.9814	0.9782	0.9780	0.9946	1.1500	1.2969	0.9941	0.9967
	3	0.9918	1.0053	1.2450	1.7437	1.0029	1.0093	0.9855	0.9924	1.3665	1.7535	0.9938	0.9924	0.9791	0.9882	1.3334	1.7560	0.9902	0.9885
	5	0.9980	1.0025	1.2024	1.9072	1.0018	1.0053	0.9954	0.9978	1.4197	1.9503	0.9982	0.9983	0.9989	1.0009	1.3327	1.9418	1.0002	1.0016
	AVG	0.9906	1.0045	1.1945	1.6526	1.0024	1.0084	0.9812	0.9891	1.3212	1.6734	0.9911	0.9896	0.9853	0.9946	1.2720	1.6649	0.9948	0.9956
0.80	1	0.9798	1.0083	1.0310	1.0846	1.0045	1.0150	0.9803	0.9927	1.0448	1.0953	0.9924	0.9932	0.9807	0.9987	1.0339	1.0838	0.9975	1.0012
	3	0.9690	1.0135	1.1024	1.2625	1.0065	1.0236	0.9569	0.9811	1.1180	1.2620	0.9836	0.9819	0.9632	0.9908	1.1135	1.2614	0.9902	0.9919
	5	0.9698	1.0143	1.1585	1.4414	1.0084	1.0292	0.9613	0.9813	1.2078	1.4530	0.9838	0.9833	0.9685	0.9949	1.1813	1.4419	0.9935	1.0008
	AVG	0.9729	1.0120	1.0973	1.2628	1.0065	1.0226	0.9662	0.9850	1.1235	1.2701	0.9866	0.9861	0.9708	0.9948	1.1096	1.2624	0.9937	0.9980
0.85	1	0.9785	1.0073	1.0165	1.0545	1.0039	1.0140	0.9814	0.9940	1.0278	1.0645	0.9936	0.9949	0.9802	0.9972	1.0196	1.0549	0.9965	0.9990
	3	0.9622	1.0134	1.0649	1.1751	1.0062	1.0247	0.9520	0.9805	1.0702	1.1728	0.9828	0.9822	0.9587	0.9913	1.0696	1.1727	0.9904	0.9936
	5	0.9583	1.0156	1.1116	1.3092	1.0085	1.0327	0.9433	0.9711	1.1299	1.3066	0.9740	0.9732	0.9566	0.9933	1.1185	1.3045	0.9914	1.0005
	AVG	0.9663	1.0121	1.0643	1.1796	1.0062	1.0238	0.9589	0.9819	1.0760	1.1813	0.9835	0.9834	0.9652	0.9939	1.0692	1.1774	0.9928	0.9977
0.90	1	0.9771	1.0068	1.0032	1.0267	1.0038	1.0140	0.9815	0.9947	1.0118	1.0352	0.9939	0.9951	0.9793	0.9977	1.0060	1.0277	0.9969	0.9999
	3	0.9541	1.0155	1.0252	1.0904	1.0080	1.0301	0.9478	0.9806	1.0240	1.0873	0.9821	0.9824	0.9533	0.9953	1.0262	1.0882	0.9925	0.9991
	5	0.9441	1.0210	1.0536	1.1708	1.0110	1.0421	0.9039	0.9396	1.0254	1.1324	0.9486	0.9420	0.9421	0.9899	1.0509	1.1640	0.9874	0.9971
	AVG	0.9584	1.0144	1.0273	1.0960	1.0076	1.0287	0.9444	0.9716	1.0204	1.0850	0.9749	0.9732	0.9582	0.9943	1.0277	1.0933	0.9923	0.9987
0.95	1	0.9771	1.0074	0.9918	1.0023	1.0039	1.0149	0.9803	0.9939	0.9956	1.0062	0.9933	0.9938	0.9790	0.9989	0.9932	1.0029	0.9973	1.0010
	3	0.9458	1.0199	0.9848	1.0104	1.0103	1.0380	0.9446	0.9801	0.9807	1.0071	0.9813	0.9811	0.9473	0.9920	0.9842	1.0095	0.9901	0.9964
	5	0.9259	1.0279	0.9867	1.0307	1.0145	1.0553	0.8619	0.9097	0.9224	0.9648	0.9262	0.9139	0.9227	0.9879	0.9802	1.0243	0.9846	0.9968
	AVG	0.9496	1.0184	0.9878	1.0145	1.0096	1.0361	0.9289	0.9612	0.9662	0.9927	0.9669	0.9629	0.9497	0.9929	0.9859	1.0122	0.9907	0.9981
0.96	1	0.9773	1.0069	0.9896	0.9976	1.0037	1.0145	0.9798	0.9932	0.9921	1.0000	0.9928	0.9931	0.9789	0.9985	0.9905	0.9979	0.9972	1.0009
	3	0.9441	1.0204	0.9765	0.9947	1.0104	1.0388	0.9438	0.9786	0.9725	0.9914	0.9801	0.9792	0.9456	0.9907	0.9755	0.9937	0.9888	0.9946
	5	0.9214	1.0310	0.9721	1.0023	1.0158	1.0595	0.8651	0.9161	0.9147	0.9453	0.9307	0.9201	0.9172	0.9863	0.9647	0.9954	0.9829	0.9944
	AVG	0.9476	1.0194	0.9794	0.9982	1.0100	1.0376	0.9296	0.9626	0.9598	0.9789	0.9679	0.9641	0.9472	0.9918	0.9769	0.9957	0.9896	0.9966
0.97	1	0.9775	1.0067	0.9871	0.9926	1.0033	1.0139	0.9790	0.9922	0.9884	0.9934	0.9923	0.9926	0.9784	0.9977	0.9874	0.9924	0.9966	1.0001
	3	0.9419	1.0211	0.9675	0.9784	1.0103	1.0398	0.9426	0.9776	0.9641	0.9753	0.9796	0.9787	0.9431	0.9885	0.9660	0.9770	0.9875	0.9927
	5	0.9161	1.0331	0.9563	0.9731	1.0164	1.0627	0.8746	0.9290	0.9134	0.9320	0.9401	0.9334	0.9104	0.9835	0.9478	0.9651	0.9814	0.9930
	AVG	0.9452	1.0203	0.9703	0.9814	1.0100	1.0388	0.9321	0.9663	0.9553	0.9669	0.9707	0.9682	0.9440	0.9899	0.9671	0.9782	0.9885	0.9953
0.98	1	0.9773	1.0064	0.9840	0.9867	1.0031	1.0138	0.9779	0.9910	0.9844	0.9861	0.9914	0.9911	0.9773	0.9969	0.9835	0.9861	0.9959	0.9990
	3	0.9385	1.0212	0.9565	0.9602	1.0108	1.0414	0.9404	0.9764	0.9547	0.9579	0.9785	0.9776	0.9394	0.9866	0.9551	0.9587	0.9864	0.9916
	5	0.9093	1.0337	0.9379	0.9418	1.0174	1.0660	0.8838	0.9435	0.9109	0.9173	0.9520	0.9487	0.9016	0.9800	0.9283	0.9320	0.9779	0.9896
	AVG	0.9417	1.0204	0.9595	0.9629	1.0104	1.0404	0.9340	0.9703	0.9500	0.9538	0.9740	0.9725	0.9394	0.9878	0.9556	0.9589	0.9867	0.9934
0.99	1	0.9761	1.0063	0.9796	0.9792	1.0030	1.0143	0.9759	0.9910	0.9791	0.9775	0.9914	0.9918	0.9753	0.9959	0.9784	0.9785	0.9955	0.9986
	3	0.9330	1.0213	0.9423	0.9388	1.0107	1.0433	0.9364	0.9741	0.9433	0.9386	0.9768	0.9757	0.9341	0.9857	0.9421	0.9380	0.9855	0.9919
	5	0.8998	1.0341	0.9146	0.9067	1.0173	1.0695	0.8845	0.9502	0.8982	0.8924	0.9570	0.9567	0.8898	0.9739	0.9039	0.8950	0.9732	0.9845
	AVG	0.9363	1.0206	0.9455	0.9416	1.0103	1.0424	0.9323	0.9718	0.9402	0.9362	0.9751	0.9747	0.9331	0.9852	0.9415	0.9372	0.9847	0.9917

Table 1: (Continued)

Part B: $T = 100$

		$N(0,1)$						$t(3)$						$t(4)$					
α	h	PMME	FPMME1	FPMME2	RW	COMB	LADE	PMME	FPMME1	FPMME2	RW	COMB	LADE	PMME	FPMME1	FPMME2	RW	COMB	LADE
0.50	1	0.9925	1.0018	1.1521	1.3308	1.0007	1.0042	0.9929	0.9991	1.2030	1.3295	0.9980	0.9992	0.9922	0.9981	1.1630	1.3076	0.9979	0.9984
	3	0.9978	1.0018	1.2382	1.7489	1.0009	1.0031	0.9960	0.9986	1.3427	1.7262	0.9988	0.9991	0.9973	1.0005	1.3290	1.7779	0.9997	1.0007
	5	0.9988	1.0008	1.1996	1.9257	1.0005	1.0015	0.9951	0.9961	1.3118	1.8433	0.9974	0.9960	0.9990	1.0003	1.3035	1.9526	1.0001	1.0005
	AVG	0.9964	1.0015	1.1966	1.6685	1.0007	1.0029	0.9947	0.9979	1.2858	1.6330	0.9981	0.9981	0.9962	0.9996	1.2652	1.6794	0.9992	0.9999
0.80	1	0.9924	1.0017	1.0394	1.0980	1.0007	1.0042	0.9878	0.9932	1.0473	1.0930	0.9931	0.9930	0.9936	0.9994	1.0486	1.0988	0.9987	0.9998
	3	0.9866	1.0055	1.1108	1.2876	1.0028	1.0108	0.9830	0.9925	1.1318	1.2773	0.9937	0.9936	0.9869	1.0011	1.1263	1.2943	0.9993	1.0033
	5	0.9851	1.0065	1.1616	1.4642	1.0040	1.0138	0.9801	0.9888	1.1990	1.4652	0.9911	0.9896	0.9824	0.9977	1.1809	1.4704	0.9970	1.0001
	AVG	0.9880	1.0046	1.1039	1.2833	1.0025	1.0096	0.9836	0.9915	1.1260	1.2785	0.9926	0.9921	0.9876	0.9994	1.1186	1.2878	0.9983	1.0011
0.85	1	0.9925	1.0015	1.0254	1.0679	1.0004	1.0037	0.9866	0.9914	1.0274	1.0612	0.9918	0.9914	0.9934	0.9992	1.0330	1.0695	0.9986	0.9997
	3	0.9844	1.0054	1.0784	1.2046	1.0022	1.0102	0.9804	0.9914	1.0910	1.1968	0.9925	0.9920	0.9849	1.0000	1.0880	1.2095	0.9982	1.0020
	5	0.9805	1.0075	1.1214	1.3381	1.0043	1.0160	0.9775	0.9894	1.1491	1.3456	0.9910	0.9905	0.9759	0.9966	1.1303	1.3399	0.9957	0.9995
	AVG	0.9858	1.0048	1.0751	1.2035	1.0023	1.0100	0.9815	0.9907	1.0892	1.2012	0.9918	0.9913	0.9847	0.9986	1.0838	1.2063	0.9975	1.0004
0.90	1	0.9924	1.0015	1.0132	1.0395	1.0005	1.0039	0.9855	0.9897	1.0095	1.0317	0.9906	0.9898	0.9925	0.9985	1.0178	1.0407	0.9984	0.9994
	3	0.9816	1.0048	1.0447	1.1227	1.0021	1.0107	0.9771	0.9904	1.0500	1.1172	0.9912	0.9906	0.9823	0.9999	1.0497	1.1266	0.9981	1.0028
	5	0.9747	1.0068	1.0729	1.2065	1.0035	1.0161	0.9737	0.9893	1.0897	1.2152	0.9902	0.9907	0.9680	0.9928	1.0732	1.2035	0.9922	0.9950
	AVG	0.9829	1.0044	1.0436	1.1229	1.0020	1.0102	0.9788	0.9898	1.0497	1.1214	0.9907	0.9904	0.9809	0.9971	1.0469	1.1236	0.9962	0.9991
0.95	1	0.9918	1.0019	1.0024	1.0125	1.0009	1.0048	0.9847	0.9887	0.9943	1.0047	0.9900	0.9885	0.9904	0.9976	1.0022	1.0119	0.9981	0.9991
	3	0.9764	1.0064	1.0097	1.0419	1.0032	1.0141	0.9706	0.9879	1.0074	1.0376	0.9892	0.9880	0.9773	0.9990	1.0113	1.0447	0.9975	1.0027
	5	0.9658	1.0085	1.0183	1.0731	1.0041	1.0200	0.9640	0.9863	1.0217	1.0779	0.9873	0.9887	0.9566	0.9884	1.0111	1.0651	0.9884	0.9911
	AVG	0.9780	1.0056	1.0101	1.0425	1.0027	1.0130	0.9731	0.9876	1.0078	1.0401	0.9888	0.9884	0.9748	0.9950	1.0082	1.0406	0.9947	0.9976
0.96	1	0.9916	1.0028	1.0002	1.0074	1.0018	1.0066	0.9846	0.9888	0.9917	0.9998	0.9900	0.9885	0.9900	0.9974	0.9992	1.0063	0.9979	0.9987
	3	0.9746	1.0078	1.0021	1.0258	1.0041	1.0164	0.9686	0.9862	0.9985	1.0214	0.9879	0.9860	0.9754	0.9981	1.0033	1.0281	0.9967	1.0016
	5	0.9632	1.0102	1.0063	1.0464	1.0050	1.0227	0.9606	0.9849	1.0069	1.0495	0.9862	0.9873	0.9532	0.9876	0.9976	1.0370	0.9877	0.9905
	AVG	0.9765	1.0069	1.0029	1.0265	1.0036	1.0152	0.9713	0.9866	0.9990	1.0236	0.9880	0.9873	0.9729	0.9944	1.0000	1.0238	0.9941	0.9969
0.97	1	0.9915	1.0030	0.9978	1.0023	1.0017	1.0066	0.9846	0.9890	0.9894	0.9952	0.9902	0.9887	0.9896	0.9977	0.9962	1.0010	0.9980	0.9990
	3	0.9723	1.0086	0.9940	1.0096	1.0047	1.0177	0.9661	0.9841	0.9893	1.0048	0.9868	0.9844	0.9729	0.9966	0.9946	1.0113	0.9956	0.9999
	5	0.9600	1.0125	0.9934	1.0196	1.0066	1.0265	0.9563	0.9828	0.9913	1.0203	0.9843	0.9849	0.9489	0.9863	0.9831	1.0085	0.9864	0.9888
	AVG	0.9746	1.0080	0.9951	1.0105	1.0043	1.0169	0.9690	0.9853	0.9900	1.0068	0.9871	0.9860	0.9705	0.9935	0.9913	1.0069	0.9933	0.9959
0.98	1	0.9910	1.0031	0.9951	0.9973	1.0020	1.0074	0.9845	0.9893	0.9875	0.9911	0.9906	0.9891	0.9892	0.9975	0.9932	0.9959	0.9978	0.9988
	3	0.9691	1.0094	0.9848	0.9927	1.0050	1.0193	0.9631	0.9822	0.9796	0.9875	0.9853	0.9820	0.9695	0.9953	0.9850	0.9938	0.9951	0.9995
	5	0.9556	1.0139	0.9790	0.9920	1.0071	1.0291	0.9504	0.9799	0.9743	0.9893	0.9822	0.9824	0.9430	0.9841	0.9666	0.9787	0.9849	0.9874
	AVG	0.9719	1.0088	0.9863	0.9940	1.0047	1.0186	0.9660	0.9838	0.9805	0.9893	0.9860	0.9845	0.9672	0.9923	0.9816	0.9895	0.9926	0.9952
0.99	1	0.9900	1.0040	0.9918	0.9918	1.0023	1.0083	0.9844	0.9898	0.9858	0.9870	0.9914	0.9900	0.9882	0.9981	0.9898	0.9905	0.9980	0.9996
	3	0.9643	1.0108	0.9733	0.9736	1.0057	1.0219	0.9593	0.9813	0.9685	0.9681	0.9841	0.9808	0.9647	0.9951	0.9734	0.9739	0.9941	0.9984
	5	0.9485	1.0160	0.9608	0.9614	1.0076	1.0325	0.9415	0.9765	0.9539	0.9542	0.9782	0.9780	0.9345	0.9814	0.9469	0.9459	0.9831	0.9861
	AVG	0.9676	1.0103	0.9753	0.9756	1.0052	1.0209	0.9617	0.9825	0.9694	0.9698	0.9846	0.9829	0.9625	0.9915	0.9700	0.9701	0.9917	0.9947

Table 1: (Continued)

Part C: $T = 200$

		$N(0,1)$						$t(3)$						$t(4)$					
α	h	PMME	FPMME1	FPMME2	RW	COMB	LADE	PMME	FPMME1	FPMME2	RW	COMB	LADE	PMME	FPMME1	FPMME2	RW	COMB	LADE
0.50	1	0.9958	1.0004	1.1556	1.3303	1.0002	1.0018	0.9953	0.9979	1.2041	1.3514	0.9982	0.9979	0.9964	0.9999	1.1677	1.3248	0.9995	1.0004
	3	0.9977	0.9999	1.2595	1.7824	0.9998	1.0003	0.9978	0.9993	1.2830	1.6684	0.9994	0.9994	0.9986	1.0002	1.3057	1.7672	1.0000	1.0006
	5	0.9998	1.0003	1.2072	1.9405	1.0003	1.0007	0.9998	1.0001	1.3822	1.9872	1.0000	1.0001	0.9997	1.0000	1.2930	1.9308	1.0000	1.0002
	AVG	0.9978	1.0002	1.2074	1.6844	1.0001	1.0009	0.9976	0.9991	1.2898	1.6690	0.9992	0.9991	0.9982	1.0000	1.2555	1.6743	0.9998	1.0004
0.80	1	0.9975	1.0017	1.0528	1.1084	1.0011	1.0037	0.9970	0.9988	1.0628	1.1186	0.9988	0.9990	0.9971	1.0001	1.0505	1.1059	0.9998	1.0010
	3	0.9938	1.0017	1.1331	1.3170	1.0007	1.0038	0.9928	0.9979	1.1299	1.2843	0.9979	0.9981	0.9942	1.0002	1.1308	1.3085	0.9994	1.0013
	5	0.9943	1.0023	1.1889	1.5075	1.0012	1.0048	0.9931	0.9963	1.2237	1.5050	0.9967	0.9962	0.9942	0.9989	1.1995	1.4910	0.9984	0.9991
	AVG	0.9952	1.0019	1.1249	1.3110	1.0010	1.0041	0.9943	0.9977	1.1388	1.3026	0.9978	0.9978	0.9952	0.9997	1.1269	1.3018	0.9992	1.0005
0.85	1	0.9981	1.0021	1.0387	1.0791	1.0014	1.0043	0.9969	0.9986	1.0448	1.0876	0.9986	0.9987	0.9972	1.0001	1.0360	1.0771	0.9997	1.0010
	3	0.9931	1.0026	1.0997	1.2323	1.0013	1.0055	0.9912	0.9974	1.0957	1.2099	0.9974	0.9976	0.9933	0.9998	1.0952	1.2257	0.9990	1.0009
	5	0.9922	1.0031	1.1519	1.3839	1.0015	1.0064	0.9907	0.9954	1.1714	1.3805	0.9959	0.9953	0.9924	0.9988	1.1573	1.3723	0.9980	0.9991
	AVG	0.9945	1.0026	1.0968	1.2318	1.0014	1.0054	0.9929	0.9971	1.1040	1.2260	0.9973	0.9972	0.9943	0.9996	1.0962	1.2250	0.9989	1.0003
0.90	1	0.9988	1.0022	1.0246	1.0511	1.0015	1.0044	0.9966	0.9981	1.0276	1.0576	0.9985	0.9985	0.9972	1.0000	1.0219	1.0494	0.9997	1.0008
	3	0.9924	1.0027	1.0642	1.1486	1.0014	1.0061	0.9892	0.9967	1.0603	1.1354	0.9969	0.9972	0.9924	0.9997	1.0597	1.1443	0.9989	1.0010
	5	0.9898	1.0037	1.1044	1.2519	1.0017	1.0077	0.9883	0.9940	1.1147	1.2513	0.9945	0.9939	0.9900	0.9988	1.1077	1.2464	0.9978	0.9997
	AVG	0.9937	1.0029	1.0644	1.1505	1.0015	1.0061	0.9914	0.9963	1.0675	1.1481	0.9966	0.9965	0.9932	0.9995	1.0631	1.1467	0.9988	1.0005
0.95	1	0.9987	1.0017	1.0098	1.0232	1.0010	1.0035	0.9962	0.9972	1.0113	1.0274	0.9981	0.9976	0.9967	0.9994	1.0078	1.0220	0.9992	0.9999
	3	0.9913	1.0038	1.0267	1.0653	1.0020	1.0077	0.9873	0.9960	1.0237	1.0609	0.9967	0.9975	0.9908	0.9996	1.0236	1.0637	0.9987	1.0010
	5	0.9867	1.0050	1.0470	1.1136	1.0022	1.0098	0.9860	0.9934	1.0529	1.1175	0.9936	0.9934	0.9862	0.9983	1.0484	1.1141	0.9974	1.0002
	AVG	0.9922	1.0035	1.0278	1.0674	1.0017	1.0070	0.9898	0.9955	1.0293	1.0686	0.9961	0.9962	0.9912	0.9991	1.0266	1.0666	0.9984	1.0004
0.96	1	0.9985	1.0016	1.0067	1.0173	1.0010	1.0034	0.9962	0.9973	1.0081	1.0211	0.9981	0.9977	0.9965	0.9993	1.0051	1.0164	0.9992	1.0000
	3	0.9908	1.0043	1.0188	1.0486	1.0022	1.0084	0.9871	0.9965	1.0162	1.0459	0.9970	0.9982	0.9901	0.9992	1.0161	1.0474	0.9984	1.0006
	5	0.9858	1.0053	1.0342	1.0853	1.0022	1.0103	0.9852	0.9929	1.0389	1.0894	0.9933	0.9929	0.9852	0.9984	1.0350	1.0867	0.9974	1.0004
	AVG	0.9917	1.0037	1.0199	1.0504	1.0018	1.0074	0.9895	0.9956	1.0211	1.0521	0.9961	0.9963	0.9906	0.9990	1.0187	1.0502	0.9983	1.0003
0.97	1	0.9981	1.0019	1.0037	1.0113	1.0012	1.0040	0.9961	0.9973	1.0051	1.0147	0.9980	0.9976	0.9963	0.9989	1.0024	1.0107	0.9990	0.9996
	3	0.9903	1.0051	1.0108	1.0317	1.0030	1.0101	0.9870	0.9964	1.0085	1.0308	0.9970	0.9984	0.9891	0.9993	1.0084	1.0310	0.9983	1.0007
	5	0.9848	1.0064	1.0208	1.0569	1.0030	1.0122	0.9838	0.9919	1.0237	1.0602	0.9926	0.9920	0.9841	0.9981	1.0212	1.0589	0.9972	1.0003
	AVG	0.9911	1.0045	1.0118	1.0333	1.0024	1.0088	0.9890	0.9952	1.0124	1.0352	0.9959	0.9960	0.9898	0.9988	1.0107	1.0335	0.9982	1.0002
0.98	1	0.9975	1.0023	1.0009	1.0052	1.0014	1.0044	0.9961	0.9972	1.0021	1.0080	0.9976	0.9970	0.9962	0.9985	1.0001	1.0051	0.9987	0.9991
	3	0.9895	1.0056	1.0025	1.0149	1.0036	1.0115	0.9867	0.9957	1.0008	1.0154	0.9963	0.9973	0.9877	0.9983	1.0005	1.0143	0.9978	0.9999
	5	0.9832	1.0076	1.0068	1.0285	1.0039	1.0145	0.9813	0.9903	1.0073	1.0296	0.9913	0.9902	0.9828	0.9978	1.0074	1.0304	0.9968	1.0001
	AVG	0.9901	1.0052	1.0034	1.0162	1.0030	1.0101	0.9880	0.9944	1.0034	1.0177	0.9951	0.9948	0.9889	0.9982	1.0027	1.0166	0.9978	0.9997
0.99	1	0.9967	1.0028	0.9983	0.9993	1.0016	1.0050	0.9956	0.9969	0.9990	1.0011	0.9971	0.9963	0.9961	0.9982	0.9983	0.9997	0.9984	0.9986
	3	0.9879	1.0063	0.9941	0.9977	1.0038	1.0126	0.9859	0.9952	0.9930	0.9990	0.9955	0.9964	0.9854	0.9966	0.9923	0.9968	0.9964	0.9978
	5	0.9800	1.0090	0.9916	0.9991	1.0048	1.0175	0.9757	0.9890	0.9888	0.9958	0.9897	0.9882	0.9797	0.9973	0.9923	0.9996	0.9959	0.9995
	AVG	0.9882	1.0060	0.9947	0.9987	1.0034	1.0117	0.9857	0.9937	0.9936	0.9986	0.9941	0.9936	0.9871	0.9974	0.9943	0.9987	0.9969	0.9986

Table 2: Empirical prediction mean-squared-errors: AR(4) model

Note: 1. Entries of this table are empirical mean-squared-errors standardized by that of the LSE-based predictor.
2. Data were generated by $(1-\alpha_1 B)(1+0.75B)(1-0.10B)(1+0.20B)y_t = u_t, u_t \sim i.i.d.N(0,1), t(3)$, or $t(4)$.
3. FPMME1 is FPMME using LAD, and FPMME2 is FPMME using $\alpha = 1$.
4. The prediction horizon is denoted by h.
5. AVG denotes the mean of the prediction mean-squared-errors over the prediction horizons.
6. The number of Monte Carlo replications is 10,000.

Part A: $T = 50$

		$N(0,1)$				$t(3)$				$t(4)$			
α_1	h	PMME	FPMME	COMB	LADE	PMME	FPMME	COMB	LADE	PMME	FPMME	COMB	LADE
0.50	1	0.9136	1.0287	1.0089	1.0419	0.8453	0.9167	0.9264	0.9194	0.9079	0.9938	0.9873	0.9996
	3	0.9559	1.0204	1.0135	1.0505	0.9502	0.9813	0.9497	0.9440	0.4820	0.4968	0.6203	0.4856
	5	0.9701	1.0129	1.0119	1.0464	0.8360	0.8313	0.7948	0.7210	0.9404	0.9669	0.9593	0.9540
	AVG	0.9465	1.0207	1.0114	1.0463	0.8772	0.9098	0.8903	0.8615	0.7768	0.8192	0.8556	0.8131
0.80	1	0.9100	1.0274	1.0079	1.0400	0.8346	0.9017	0.9186	0.9041	0.9079	0.9906	0.9855	0.9959
	3	0.9362	1.0266	1.0127	1.0517	0.8467	0.9028	0.8935	0.8860	0.5123	0.5378	0.6575	0.5363
	5	0.9518	1.0201	1.0136	1.0537	0.7879	0.7955	0.8028	0.7243	0.9251	0.9644	0.9559	0.9524
	AVG	0.9327	1.0247	1.0114	1.0485	0.8231	0.8667	0.8716	0.8381	0.7818	0.8309	0.8663	0.8282
0.85	1	0.9092	1.0300	1.0092	1.0427	0.8315	0.8986	0.9157	0.9009	0.9079	0.9930	0.9867	0.9987
	3	0.9303	1.0307	1.0145	1.0568	0.8492	0.9066	0.8997	0.8911	0.5284	0.5586	0.6688	0.5567
	5	0.9434	1.0255	1.0156	1.0607	0.7785	0.7978	0.8036	0.7254	0.9196	0.9657	0.9598	0.9599
	AVG	0.9276	1.0287	1.0131	1.0534	0.8197	0.8677	0.8730	0.8391	0.7853	0.8391	0.8718	0.8384
0.90	1	0.9084	1.0318	1.0100	1.0443	0.8276	0.8974	0.9121	0.8997	0.9077	0.9953	0.9876	1.0014
	3	0.9235	1.0329	1.0148	1.0582	0.8632	0.9241	0.9148	0.9102	0.5477	0.5832	0.6832	0.5816
	5	0.9319	1.0290	1.0160	1.0641	0.7905	0.8282	0.8237	0.7551	0.9146	0.9720	0.9675	0.9758
	AVG	0.9213	1.0312	1.0136	1.0555	0.8271	0.8832	0.8835	0.8550	0.7900	0.8502	0.8794	0.8529
0.95	1	0.9078	1.0296	1.0086	1.0418	0.8245	0.8949	0.9101	0.8972	0.9066	0.9947	0.9868	1.0006
	3	0.9159	1.0360	1.0159	1.0629	0.8863	0.9511	0.9389	0.9380	0.5736	0.6175	0.7046	0.6185
	5	0.9165	1.0373	1.0194	1.0775	0.8210	0.8795	0.8792	0.8384	0.9073	0.9790	0.9729	0.9856
	AVG	0.9134	1.0343	1.0146	1.0607	0.8439	0.9085	0.9094	0.8912	0.7958	0.8637	0.8881	0.8682
0.96	1	0.9074	1.0312	1.0095	1.0438	0.8240	0.8953	0.9103	0.8977	0.9056	0.9952	0.9869	1.0011
	3	0.9140	1.0382	1.0170	1.0658	0.8906	0.9577	0.9436	0.9440	0.5808	0.6263	0.7104	0.6274
	5	0.9128	1.0407	1.0213	1.0826	0.8201	0.8824	0.8882	0.8528	0.9043	0.9800	0.9725	0.9861
	AVG	0.9114	1.0367	1.0159	1.0641	0.8449	0.9118	0.9140	0.8982	0.7969	0.8672	0.8899	0.8715
0.97	1	0.9064	1.0312	1.0096	1.0442	0.8235	0.8957	0.9112	0.8982	0.9041	0.9950	0.9865	1.0009
	3	0.9118	1.0385	1.0168	1.0659	0.8938	0.9623	0.9487	0.9492	0.5890	0.6356	0.7168	0.6369
	5	0.9085	1.0420	1.0215	1.0842	0.8161	0.8810	0.8921	0.8595	0.9001	0.9814	0.9723	0.9876
	AVG	0.9089	1.0372	1.0160	1.0648	0.8445	0.9130	0.9173	0.9023	0.7977	0.8707	0.8919	0.8751
0.98	1	0.9052	1.0310	1.0095	1.0443	0.8227	0.8963	0.9119	0.8991	0.9024	0.9940	0.9860	1.0000
	3	0.9090	1.0398	1.0169	1.0664	0.8955	0.9668	0.9535	0.9543	0.5982	0.6454	0.7241	0.6475
	5	0.9032	1.0470	1.0228	1.0884	0.8097	0.8800	0.8898	0.8594	0.8938	0.9797	0.9682	0.9832
	AVG	0.9058	1.0393	1.0164	1.0664	0.8426	0.9144	0.9184	0.9043	0.7981	0.8730	0.8928	0.8769
0.99	1	0.9035	1.0307	1.0091	1.0441	0.8211	0.8955	0.9145	0.8981	0.9003	0.9917	0.9846	0.9978
	3	0.9051	1.0391	1.0164	1.0666	0.8956	0.9692	0.9562	0.9577	0.6082	0.6602	0.7337	0.6620
	5	0.8960	1.0464	1.0220	1.0892	0.8006	0.8717	0.8795	0.8468	0.8839	0.9797	0.9647	0.9794
	AVG	0.9015	1.0387	1.0158	1.0666	0.8391	0.9121	0.9167	0.9009	0.7975	0.8772	0.8943	0.8797

Table 2: (Continued)

Part B: $T = 100$

		$N(0,1)$				$t(3)$				$t(4)$			
α_1	h	PMME	FPMME	COMB	LADE	PMME	FPMME	COMB	LADE	PMME	FPMME	COMB	LADE
0.50	1	0.9610	1.0170	1.0067	1.0247	0.9582	0.9851	0.9864	0.9862	0.9578	0.9953	0.9925	0.9978
	3	0.9810	1.0069	1.0036	1.0178	0.8849	0.8951	0.9181	0.8920	0.9789	0.9933	0.9913	0.9925
	5	0.9886	1.0066	1.0043	1.0176	0.9817	0.9864	0.9790	0.9740	0.9877	0.9976	0.9941	0.9981
	AVG	0.9769	1.0102	1.0049	1.0200	0.9416	0.9555	0.9612	0.9507	0.9748	0.9954	0.9926	0.9961
0.80	1	0.9616	1.0187	1.0074	1.0264	0.9598	0.9842	0.9858	0.9853	0.9581	0.9943	0.9925	0.9965
	3	0.9730	1.0130	1.0059	1.0236	0.8656	0.8834	0.9114	0.8827	0.9691	0.9924	0.9914	0.9933
	5	0.9801	1.0105	1.0061	1.0231	0.9620	0.9729	0.9670	0.9607	0.9775	0.9958	0.9927	0.9972
	AVG	0.9716	1.0141	1.0065	1.0244	0.9291	0.9468	0.9547	0.9429	0.9682	0.9942	0.9922	0.9957
0.85	1	0.9619	1.0190	1.0075	1.0267	0.9602	0.9839	0.9856	0.9849	0.9578	0.9949	0.9929	0.9973
	3	0.9708	1.0139	1.0060	1.0244	0.8631	0.8820	0.9115	0.8816	0.9669	0.9933	0.9925	0.9959
	5	0.9766	1.0121	1.0070	1.0260	0.9558	0.9704	0.9650	0.9593	0.9740	0.9958	0.9921	0.9975
	AVG	0.9698	1.0150	1.0068	1.0257	0.9264	0.9454	0.9540	0.9419	0.9662	0.9947	0.9925	0.9969
0.90	1	0.9621	1.0185	1.0071	1.0260	0.9606	0.9847	0.9861	0.9859	0.9577	0.9949	0.9927	0.9973
	3	0.9681	1.0146	1.0064	1.0256	0.8606	0.8808	0.9137	0.8808	0.9646	0.9941	0.9935	0.9983
	5	0.9719	1.0146	1.0082	1.0297	0.9483	0.9673	0.9637	0.9559	0.9692	0.9940	0.9898	0.9946
	AVG	0.9674	1.0159	1.0072	1.0271	0.9232	0.9443	0.9545	0.9409	0.9638	0.9943	0.9920	0.9967
0.95	1	0.9621	1.0198	1.0079	1.0277	0.9603	0.9862	0.9869	0.9873	0.9580	0.9948	0.9925	0.9967
	3	0.9644	1.0173	1.0075	1.0284	0.8595	0.8839	0.9151	0.8836	0.9610	0.9947	0.9934	0.9992
	5	0.9650	1.0178	1.0083	1.0320	0.9385	0.9643	0.9623	0.9531	0.9604	0.9912	0.9870	0.9912
	AVG	0.9638	1.0183	1.0079	1.0294	0.9194	0.9448	0.9548	0.9413	0.9598	0.9936	0.9910	0.9957
0.96	1	0.9620	1.0192	1.0075	1.0270	0.9597	0.9865	0.9870	0.9876	0.9582	0.9956	0.9929	0.9976
	3	0.9632	1.0174	1.0074	1.0287	0.8597	0.8845	0.9156	0.8844	0.9598	0.9940	0.9929	0.9983
	5	0.9631	1.0183	1.0083	1.0327	0.9362	0.9628	0.9614	0.9517	0.9576	0.9915	0.9872	0.9922
	AVG	0.9628	1.0183	1.0077	1.0295	0.9185	0.9446	0.9547	0.9412	0.9585	0.9937	0.9910	0.9960
0.97	1	0.9619	1.0195	1.0077	1.0276	0.9589	0.9870	0.9873	0.9881	0.9583	0.9953	0.9927	0.9970
	3	0.9617	1.0166	1.0068	1.0276	0.8600	0.8858	0.9164	0.8856	0.9583	0.9940	0.9932	0.9993
	5	0.9607	1.0182	1.0081	1.0331	0.9336	0.9600	0.9592	0.9496	0.9542	0.9911	0.9870	0.9929
	AVG	0.9614	1.0181	1.0075	1.0294	0.9175	0.9443	0.9543	0.9411	0.9569	0.9935	0.9910	0.9964
0.98	1	0.9616	1.0195	1.0077	1.0276	0.9576	0.9867	0.9871	0.9878	0.9583	0.9948	0.9924	0.9965
	3	0.9596	1.0174	1.0073	1.0291	0.8602	0.8855	0.9160	0.8845	0.9564	0.9925	0.9922	0.9979
	5	0.9575	1.0208	1.0094	1.0367	0.9307	0.9587	0.9579	0.9483	0.9497	0.9908	0.9866	0.9929
	AVG	0.9596	1.0192	1.0081	1.0311	0.9162	0.9436	0.9537	0.9402	0.9548	0.9927	0.9904	0.9958
0.99	1	0.9611	1.0205	1.0080	1.0287	0.9555	0.9858	0.9863	0.9869	0.9574	0.9950	0.9925	0.9970
	3	0.9563	1.0188	1.0079	1.0314	0.8599	0.8853	0.9153	0.8829	0.9537	0.9921	0.9914	0.9973
	5	0.9527	1.0241	1.0109	1.0414	0.9275	0.9591	0.9580	0.9490	0.9434	0.9880	0.9848	0.9908
	AVG	0.9567	1.0211	1.0089	1.0338	0.9143	0.9434	0.9532	0.9396	0.9515	0.9917	0.9896	0.9950

Table 2: (Continued)

Part C: $T = 200$

		$N(0,1)$				$t(3)$				$t(4)$			
α_1	h	PMME	FPMME	COMB	LADE	PMME	FPMME	COMB	LADE	PMME	FPMME	COMB	LADE
0.50	1	0.9793	1.0079	1.0029	1.0115	0.9787	0.9941	0.9944	0.9949	0.9807	0.9979	0.9968	0.9991
	3	0.9911	1.0048	1.0030	1.0111	0.9875	0.9934	0.9910	0.9891	0.9918	1.0006	0.9988	1.0021
	5	0.9941	1.0013	1.0027	1.0092	0.9927	0.9947	0.9947	0.9938	0.9938	0.9993	0.9978	0.9991
	AVG	0.9882	1.0047	1.0029	1.0106	0.9863	0.9941	0.9934	0.9926	0.9888	0.9993	0.9978	1.0001
0.80	1	0.9808	1.0075	1.0026	1.0110	0.9780	0.9926	0.9935	0.9932	0.9808	0.9982	0.9970	0.9995
	3	0.9879	1.0052	1.0027	1.0110	0.9846	0.9935	0.9933	0.9934	0.9881	1.0008	0.9989	1.0028
	5	0.9925	1.0017	1.0020	1.0087	0.9890	0.9930	0.9939	0.9927	0.9924	0.9987	0.9970	0.9986
	AVG	0.9871	1.0048	1.0024	1.0102	0.9839	0.9930	0.9936	0.9931	0.9871	0.9992	0.9976	1.0003
0.85	1	0.9811	1.0079	1.0029	1.0116	0.9779	0.9926	0.9935	0.9932	0.9808	0.9980	0.9969	0.9993
	3	0.9871	1.0055	1.0028	1.0113	0.9834	0.9936	0.9937	0.9941	0.9873	1.0008	0.9987	1.0025
	5	0.9913	1.0020	1.0019	1.0091	0.9874	0.9929	0.9931	0.9921	0.9911	0.9981	0.9963	0.9978
	AVG	0.9865	1.0051	1.0025	1.0107	0.9829	0.9930	0.9934	0.9931	0.9864	0.9990	0.9973	0.9999
0.90	1	0.9814	1.0078	1.0028	1.0115	0.9780	0.9925	0.9934	0.9932	0.9808	0.9980	0.9969	0.9994
	3	0.9863	1.0070	1.0038	1.0135	0.9819	0.9931	0.9935	0.9938	0.9867	0.9994	0.9978	1.0009
	5	0.9896	1.0031	1.0027	1.0114	0.9855	0.9918	0.9917	0.9904	0.9889	0.9974	0.9957	0.9973
	AVG	0.9858	1.0060	1.0031	1.0121	0.9818	0.9925	0.9929	0.9925	0.9855	0.9983	0.9968	0.9992
0.95	1	0.9813	1.0078	1.0027	1.0113	0.9788	0.9924	0.9935	0.9933	0.9806	0.9976	0.9967	0.9990
	3	0.9858	1.0084	1.0044	1.0150	0.9804	0.9922	0.9932	0.9936	0.9862	0.9998	0.9980	1.0018
	5	0.9874	1.0052	1.0034	1.0136	0.9831	0.9909	0.9912	0.9905	0.9857	0.9961	0.9951	0.9970
	AVG	0.9848	1.0071	1.0035	1.0133	0.9808	0.9918	0.9926	0.9925	0.9842	0.9978	0.9966	0.9993
0.96	1	0.9811	1.0076	1.0026	1.0110	0.9791	0.9927	0.9936	0.9936	0.9805	0.9973	0.9964	0.9985
	3	0.9857	1.0086	1.0045	1.0153	0.9802	0.9923	0.9933	0.9938	0.9859	1.0001	0.9982	1.0023
	5	0.9869	1.0055	1.0035	1.0141	0.9822	0.9901	0.9909	0.9902	0.9850	0.9959	0.9951	0.9972
	AVG	0.9846	1.0072	1.0035	1.0135	0.9805	0.9917	0.9926	0.9925	0.9838	0.9978	0.9966	0.9993
0.97	1	0.9808	1.0075	1.0024	1.0108	0.9792	0.9929	0.9937	0.9937	0.9804	0.9973	0.9963	0.9983
	3	0.9855	1.0093	1.0050	1.0167	0.9801	0.9923	0.9932	0.9937	0.9854	0.9997	0.9979	1.0018
	5	0.9864	1.0063	1.0043	1.0163	0.9809	0.9897	0.9907	0.9901	0.9842	0.9953	0.9948	0.9968
	AVG	0.9842	1.0077	1.0039	1.0146	0.9801	0.9916	0.9925	0.9925	0.9833	0.9974	0.9963	0.9990
0.98	1	0.9803	1.0070	1.0020	1.0100	0.9791	0.9930	0.9936	0.9938	0.9802	0.9974	0.9963	0.9985
	3	0.9852	1.0094	1.0051	1.0171	0.9798	0.9918	0.9927	0.9930	0.9845	0.9986	0.9970	1.0002
	5	0.9857	1.0081	1.0054	1.0191	0.9788	0.9890	0.9900	0.9893	0.9830	0.9957	0.9950	0.9975
	AVG	0.9837	1.0082	1.0042	1.0154	0.9792	0.9913	0.9921	0.9920	0.9826	0.9972	0.9961	0.9987
0.99	1	0.9793	1.0074	1.0023	1.0107	0.9780	0.9928	0.9934	0.9935	0.9796	0.9971	0.9959	0.9979
	3	0.9838	1.0109	1.0056	1.0185	0.9791	0.9909	0.9916	0.9913	0.9823	0.9980	0.9961	0.9990
	5	0.9836	1.0110	1.0068	1.0229	0.9745	0.9875	0.9885	0.9871	0.9805	0.9962	0.9948	0.9979
	AVG	0.9822	1.0098	1.0049	1.0174	0.9772	0.9904	0.9912	0.9906	0.9808	0.9971	0.9956	0.9983

Table 3: Empirical mean-squared-errors, bias and variance

Note: 1. Data were generated by $(1-\alpha B)y_t = u_t$, where $u_t \sim i.i.d.N(0,1)$, or $t(3)$. The AR root is $\alpha = 0.5$ for the stationary case, whereas $\alpha = 1 - \frac{2.5}{T}$ for the near unit root case.

2. FPMME employs LADE.

3. The number of Monte Carlo replications is 10,000.

Part A: The case of stationarity

(a) Standard normal

	T=100				T=200			
	PMME	FPMME	LSE	LADE	PMME	FPMME	LSE	LADE
MSE	0.003872	0.008640	0.007843	0.011962	0.001901	0.004211	0.003842	0.005904
Bias	-0.006503	-0.009671	-0.010198	-0.009655	-0.003737	-0.005414	-0.005732	-0.005422
Variance	0.003830	0.008546	0.007739	0.011869	0.001887	0.004182	0.003810	0.005875

(b) t(3)

	T=100				T=200			
	PMME	FPMME	LSE	LADE	PMME	FPMME	LSE	LADE
MSE	0.003986	0.006251	0.007130	0.005868	0.002093	0.003118	0.003662	0.002754
Bias	-0.005819	-0.007168	-0.008438	-0.005162	-0.004075	-0.004740	-0.005670	-0.003079
Variance	0.003953	0.006200	0.007059	0.005842	0.002076	0.003095	0.003630	0.002744

Part B: The case of a near unit root

(a) Standard normal

	T=100				T=200			
	PMME	FPMME	LSE	LADE	PMME	FPMME	LSE	LADE
MSE	0.000637	0.001501	0.001397	0.001830	0.000178	0.000415	0.000384	0.000504
Bias	-0.011289	-0.016586	-0.016661	-0.016424	-0.006003	-0.008888	-0.008872	-0.008802
Variance	0.000509	0.001226	0.001120	0.001560	0.000142	0.000336	0.000305	0.000427

(b) t(3)

	T=100				T=200			
	PMME	FPMME	LSE	LADE	PMME	FPMME	LSE	LADE
MSE	0.000641	0.001091	0.001367	0.000851	0.000171	0.000281	0.000354	0.000204
Bias	-0.010827	-0.013696	-0.015553	-0.009727	-0.005703	-0.007225	-0.008308	-0.004773
Variance	0.000523	0.000903	0.001125	0.000757	0.000138	0.000229	0.000285	0.000181

Figure 1: Empirical probability density functions: The case of stationarity

Note: 1. Data were generated by $(1-\alpha B)y_t = u_t$, where $\alpha = 0.5$ and $u_t \sim i.i.d.N(0,1)$, or $t(3)$.
2. These figures are drawn for $\sqrt{T_1}(\widehat{\alpha}-\alpha)$ with 10,000 replications.

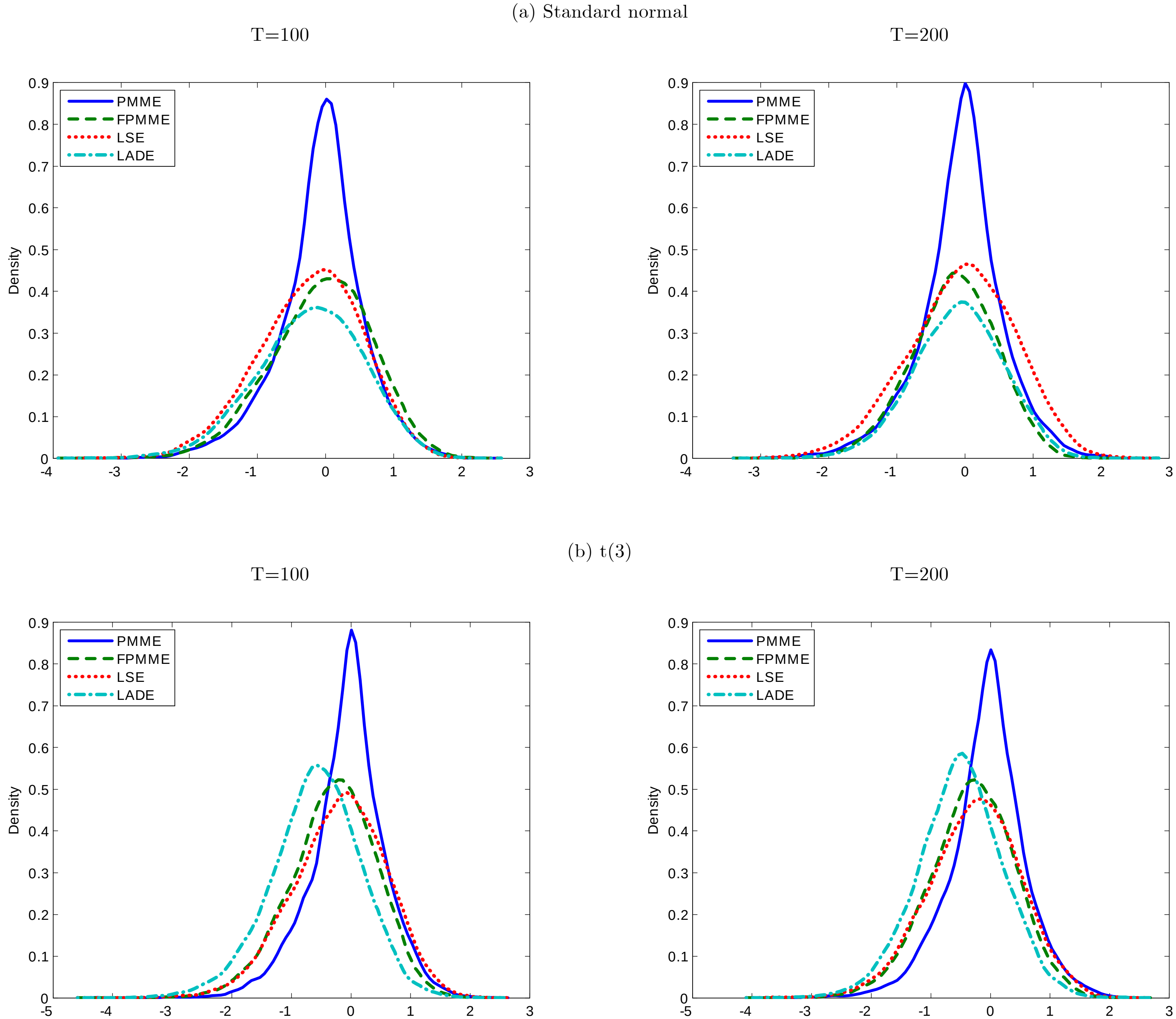
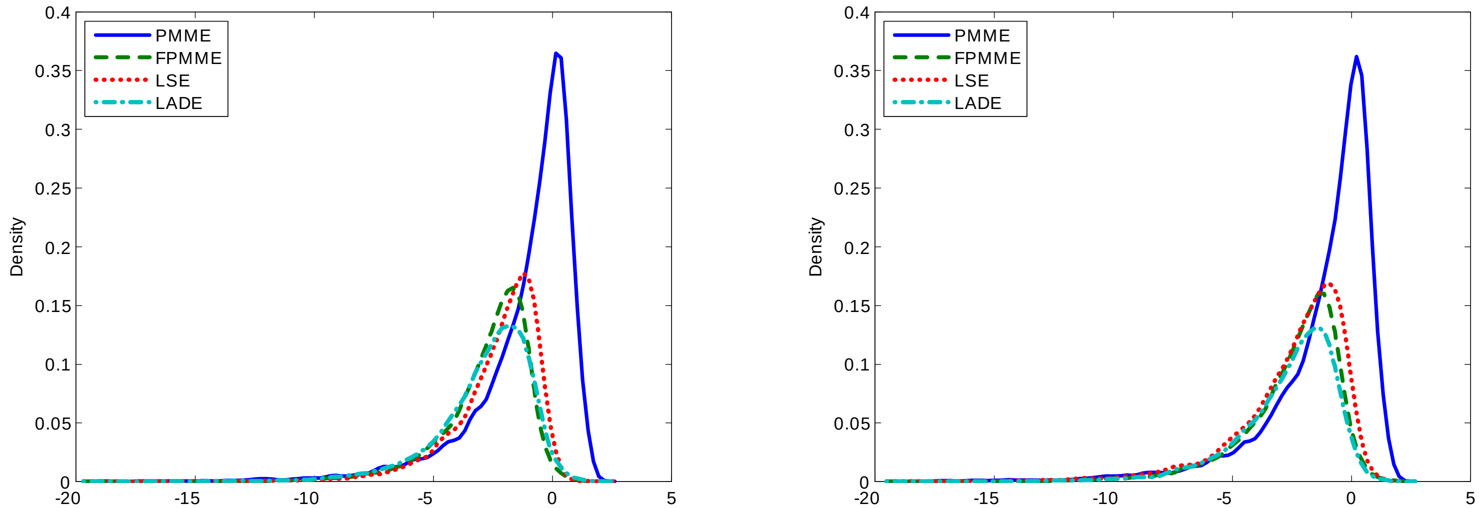


Figure 2: Empirical probability density functions: The case of a near unit root

Note: 1. Data were generated by $(1-\alpha B)y_t = u_t$, where $\alpha = 1 - \frac{2.5}{T}$ and $u_t \sim i.i.d.N(0,1)$, or $t(3)$.
2. These figures are drawn for $T_1(\widehat{\alpha}-\alpha)$ with 10,000 replications.

(a) Standard normal

T=100 T=200



(b) t(3)

T=100 T=200